Anderson Localization:

Multifractality, Symmetries, Topologies, and Electron-Electron Interaction

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Plan

- **Anderson localization: basic properties, field theory**
- **Wave function multifractality**
- **Symmetries of disordered systems**
- **Manifestations of topology in localization theory**
- **Influence of electron-electron interaction**

Anderson localization

Philip W. Anderson

**1958 "Absence of diffusion
in certain random lattices"**

sufficiently strong disorder $\longrightarrow$ quantum localization

$\longrightarrow$ eigenstates exponentially localized, no diffusion

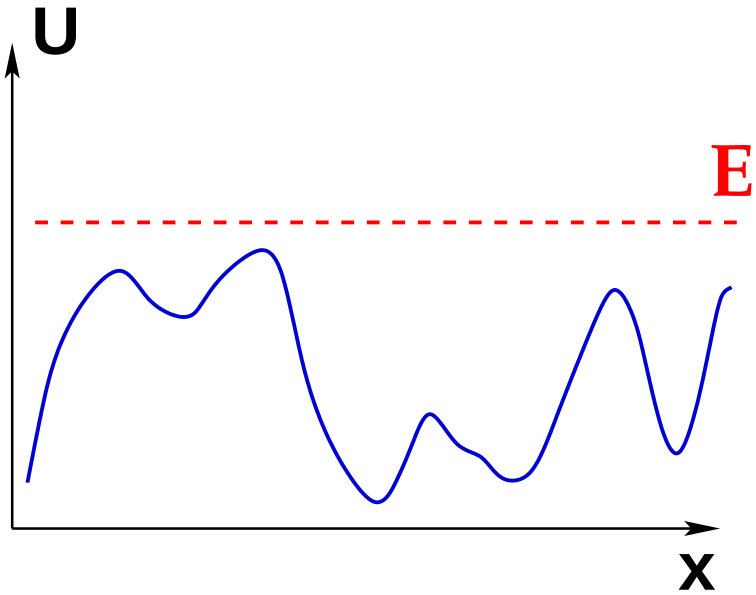
$\longrightarrow$ Anderson insulator

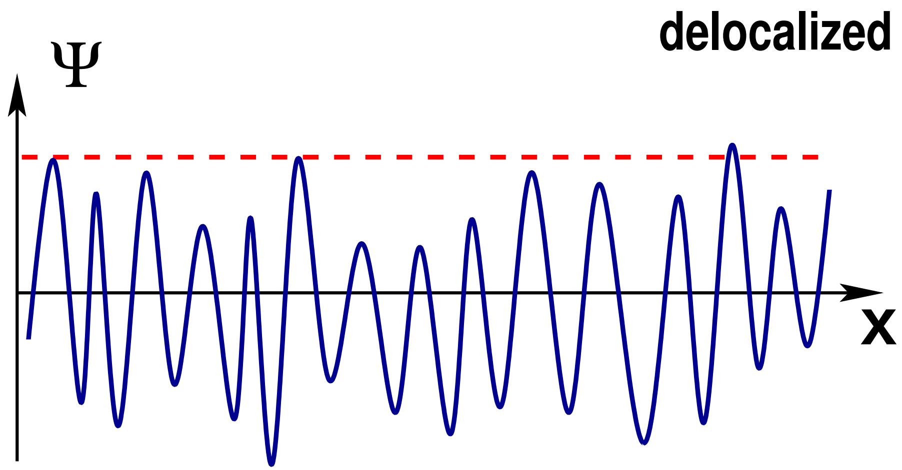
Nobel Prize 1977

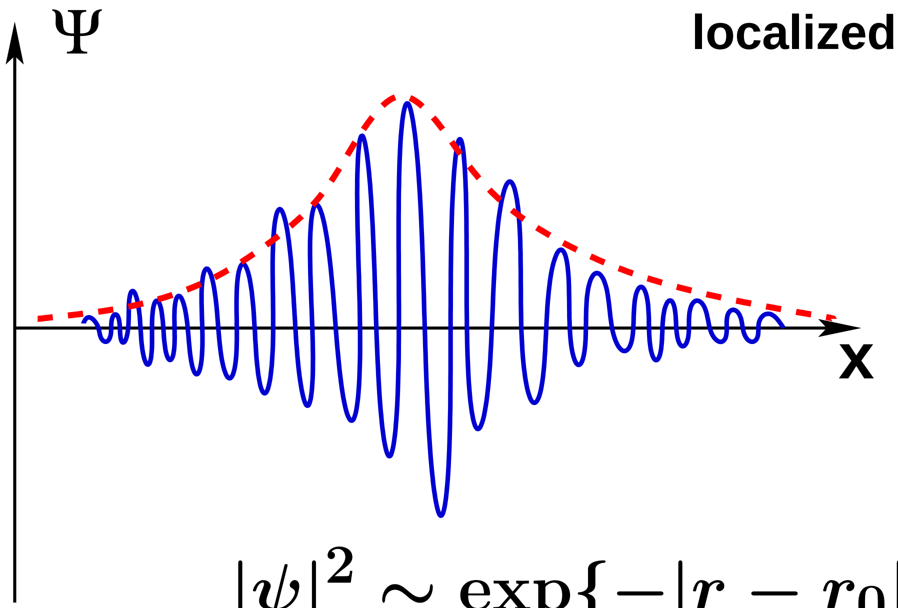
Anderson Localization: Extended and localized wave functions

Schrödinger equation
in a random potential

$$[-\hbar^2 \frac{\Delta}{2m} + U(\mathbf{r})]\psi = E\psi$$

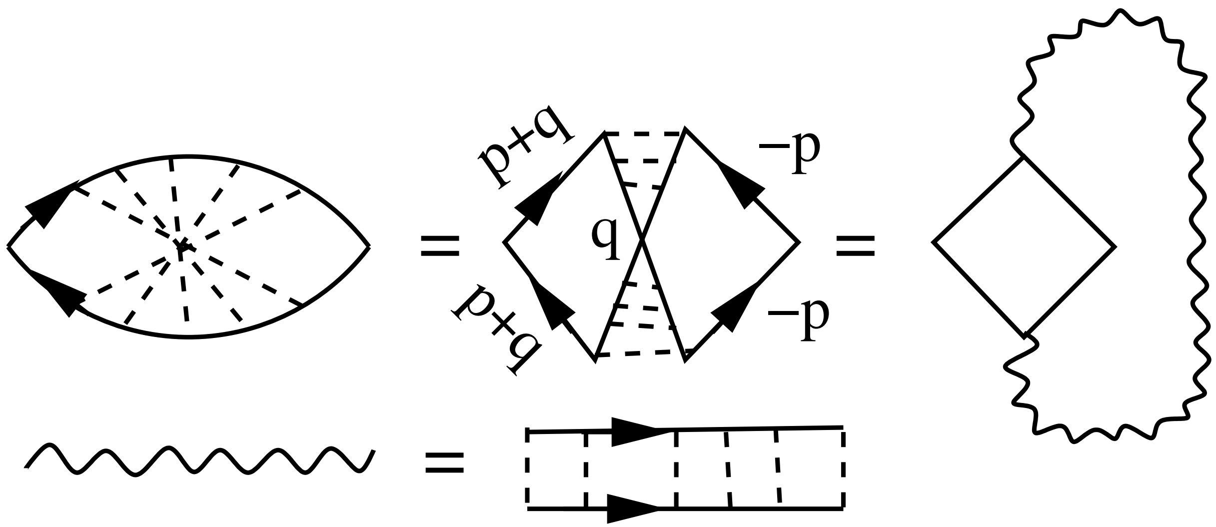


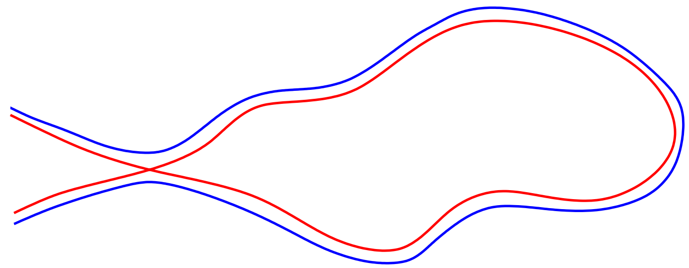




$$|\psi|^2 \sim \exp\{-|r - r_0|/\xi\}$$

Precursor of strong Anderson localization: Weak localization





Cooperon loop (interference of time-reversed paths)

$$\Delta\sigma_{\rm WL} \simeq \sigma_0 \frac{1}{\pi\nu}\int \frac{(dq)}{Dq^2 - i\omega}$$

$$\Delta\sigma_{\rm WL} = -\frac{e^2}{2\pi h}\left(\frac{\sim 1}{l} - \frac{1}{L_\omega}\right), \quad \text{3D}$$

$$L_\omega \simeq (D/\omega)^{1/2}$$

$$\Delta\sigma_{\rm WL} = -\frac{e^2}{\pi h}\ln\frac{L_\omega}{l}\ , \quad \text{2D}$$

Generally: IR cutoff

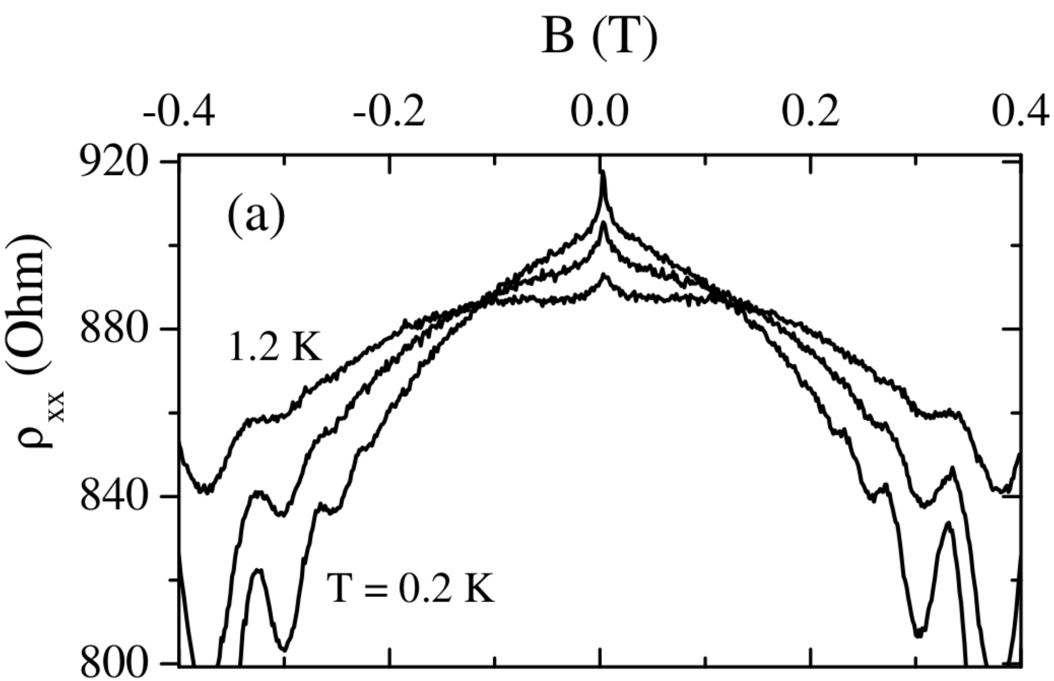
$$L_\omega \longrightarrow \min\{L_\omega, L_\phi, L, L_H\}$$

$$\Delta\sigma_{\rm WL} = -\frac{e^2}{h}L_\omega\ , \quad \text{quasi-1D}$$

$e^2/h \simeq (25\ k\Omega)^{-1}$

“conductance quantum”

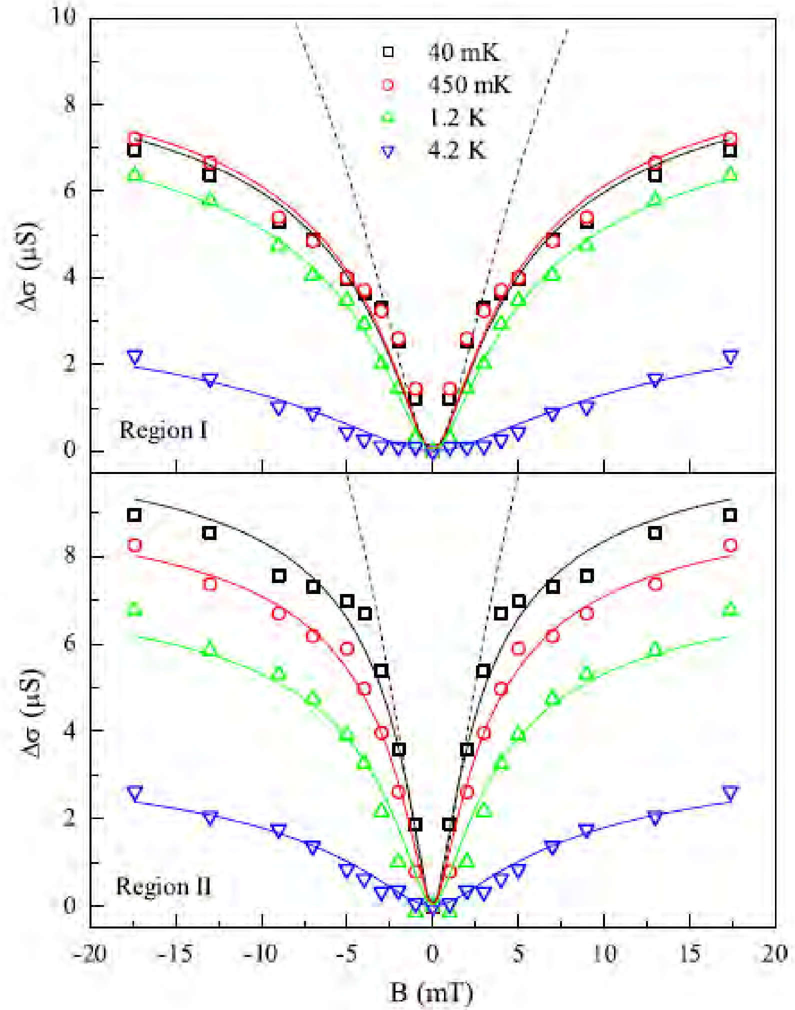
Weak localization in experiment: Magnetoresistance



Li et al. (Savchenko group), PRL'03

2D electron gas
in GaAs heterostructure

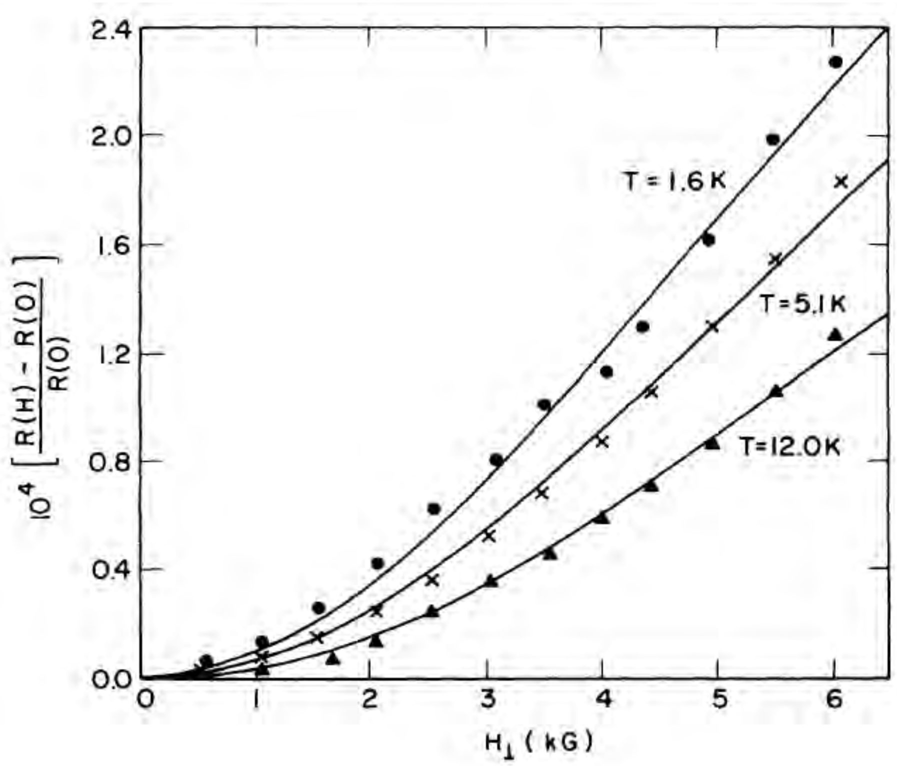
low field: weak localization

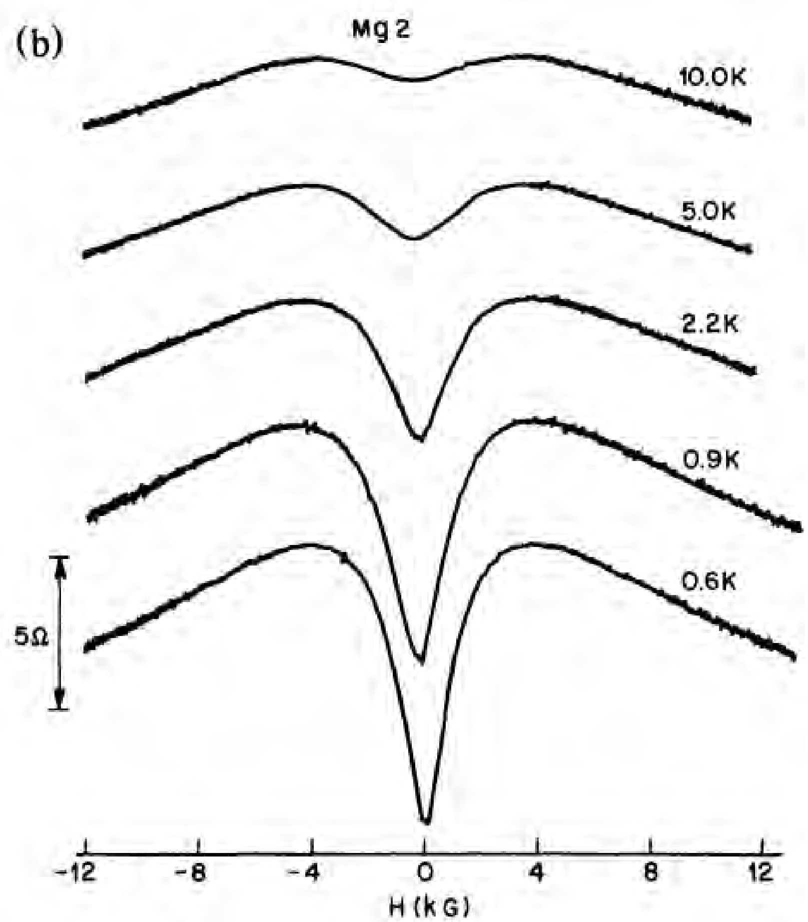


Gorbachev et al. (Savchenko group), PRL'07

weak localization in bilayer graphene

Weak localization in experiment: Magnetoresistance (cont'd)





Lin, Giordano, PRL'86

Au-Pd wires; weak antilocalization due to strong spin-orbit scattering

White, Dynes, Garno PRB'84

Mg films; weak antilocalization at lowest fields;
weak localization at stronger fields

Altshuler-Aronov-Spivak effect: $\Phi_0/2$ AB oscillations

The Aaronov–Bohm effect in disordered conductors

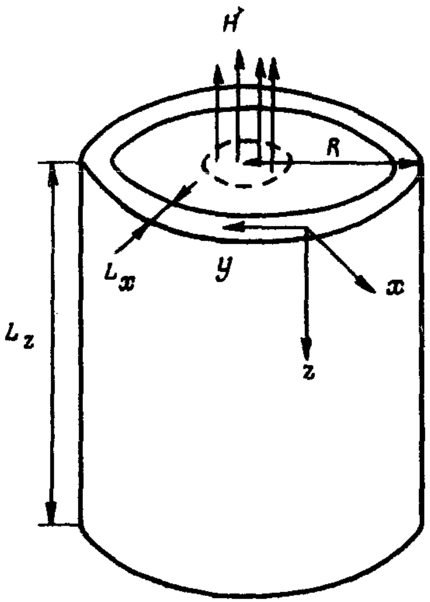
B. L. Al'tshuler, A. G. Aronov, and B. Z. Spivak
B. P. Konstantinov Institute of Nuclear Physics, USSR Academy of Sciences

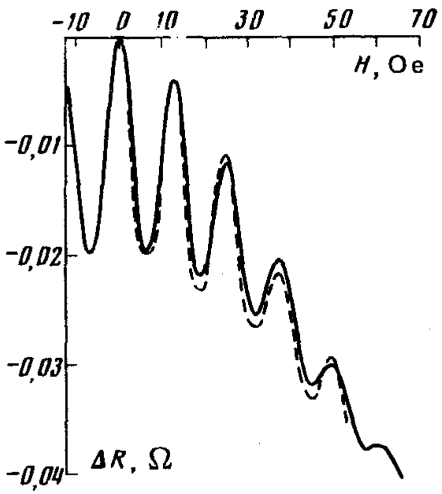
(Submitted 18 November 1980)

Pis'ma Zh. Eksp. Teor. Fiz. **33**, No. 2, 101–103 (20 January 1981)

It is shown that the Aaronov-Bohm effect, which is manifested in the oscillations of the kinetic coefficients as a function of the magnetic flux that penetrates the sample, must exist in disordered normal conductors. The period of these oscillations is $\Phi_0 = hc/2e$, i.e., it is half as large as in the ordinary Aaronov-Bohm effect.

Arkady Aronov (1939-1994)

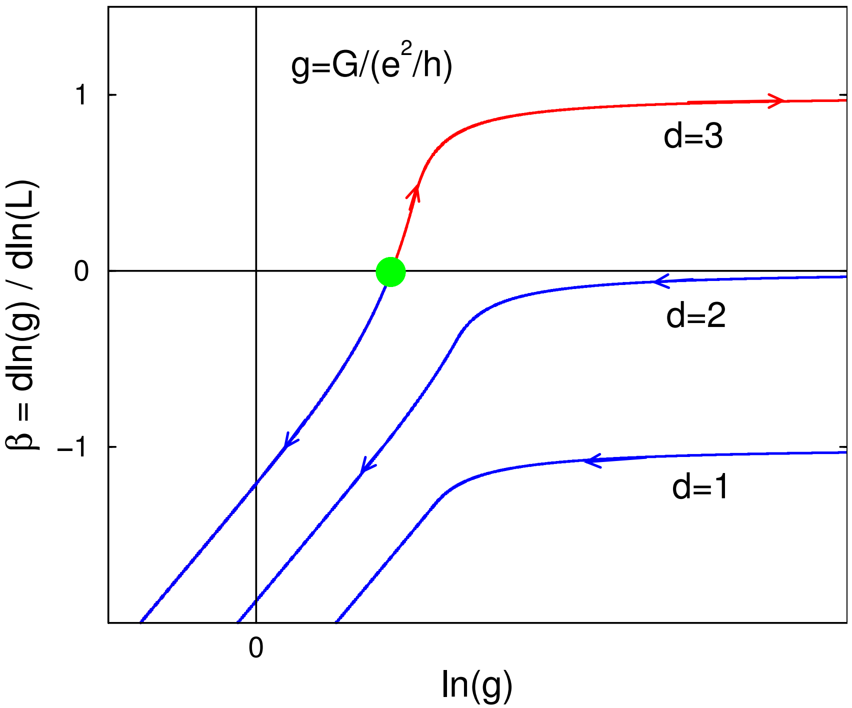




experimental observation:
Sharvin, Sharvin '81

review: Aronov, Sharvin,
Rev. Mod. Phys.'87

Anderson Insulators & Metals



Connection with scaling theory of critical phenomena: Thouless '74; Wegner '76

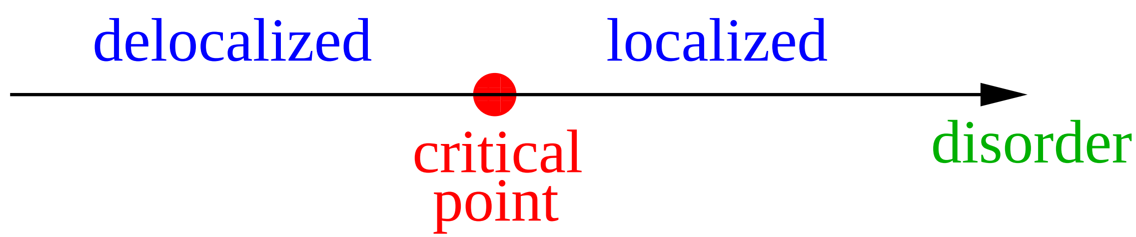
Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

scaling variable:
dimensionless conductance $g = G/(e^2/h)$

RG for field theory (σ-model)
Wegner '79

quasi-1D, 2D :
all states are localized

$d > 2$: **Anderson metal-insulator transition**



review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Field theory: non-linear σ-model

action:

$$S[Q] = \frac{\pi\nu}{4}\int d^d\mathrm{r}\ \mathrm{Tr}\,[-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(\mathrm{r}) = 1$$

Wegner'79

σ-model manifold:

e.g., "unitary" symmetry class (broken time-reversal symmetry):

- fermionic replicas: $\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$, $n\to 0$ "sphere"
- bosonic replicas: $\mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)$, $n\to 0$ "hyperboloid"

- supersymmetry (Efetov'83): $\mathrm{U}(1,1|2)/\mathrm{U}(1|1)\times\mathrm{U}(1|1)$

{"sphere" × "hyperboloid"} "dressed" by anticommuting variables

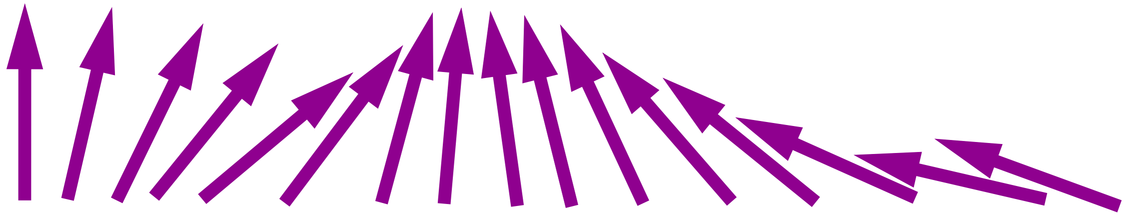
- with electron-electron interaction: Finkelstein'83

σ model: Perturbative treatment

For comparison, consider ferromagnet model in external magnetic field:

$$H[\mathrm{S}] = \int d^d r \left[\frac{\kappa}{2} (\nabla \mathrm{S}(r))^2 - \mathrm{BS}(r) \right] , \qquad \mathrm{S}^2(r) = 1$$

n-component vector σ-model



Target manifold:

sphere $S^{n-1} = O(n)/O(n-1)$

Independent degrees of freedom: transverse part $\mathrm{S}_\perp$; $\quad S_1 = (1 - \mathrm{S}_\perp^2)^{1/2}$

$$H[\mathrm{S}_\perp] = \frac{1}{2} \int d^d r \left[\kappa [\nabla \mathrm{S}_\perp(r)]^2 + B \mathrm{S}_\perp^2(r) + O(\mathrm{S}_\perp^4(r)) \right]$$

Ferromagnetic phase: broken symmetry,

Goldstone modes – spin waves

$$\langle \mathrm{S}_\perp \mathrm{S}_\perp \rangle_q \propto \frac{1}{\kappa q^2 + B}$$

Similarly

$$S[Q] = \frac{\pi\nu}{4} \int d^d r \, \mathrm{Str}[D(\nabla Q_\perp)^2 - i\omega Q_\perp^2 + O(Q_\perp^3)]$$

theory of "interacting" diffusion modes;

Goldstone mode: diffusion propagator

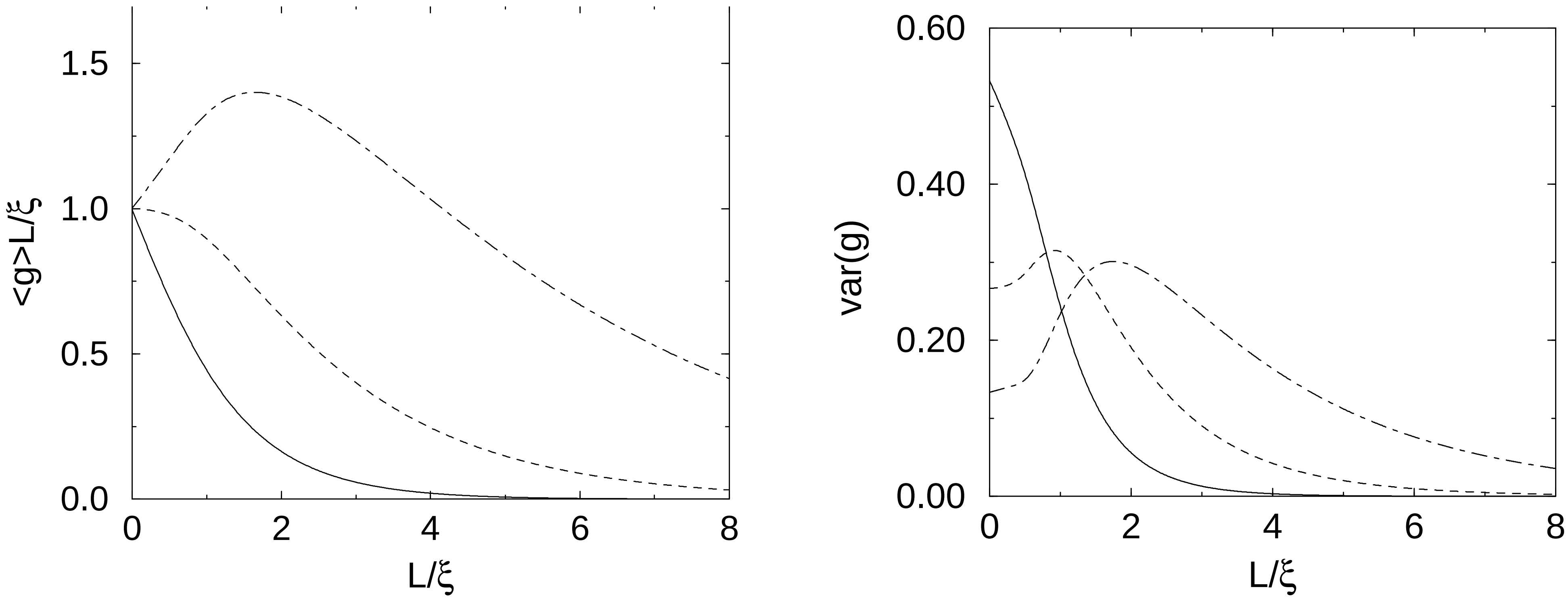
$$\langle Q_\perp Q_\perp \rangle_{q,\omega} \sim \frac{1}{\pi\nu(Dq^2 - i\omega)}$$

Quasi-1D geometry: Exact solution of the σ-model

quasi-1D geometry (many-channel wire) $\longrightarrow$ 1D σ-model

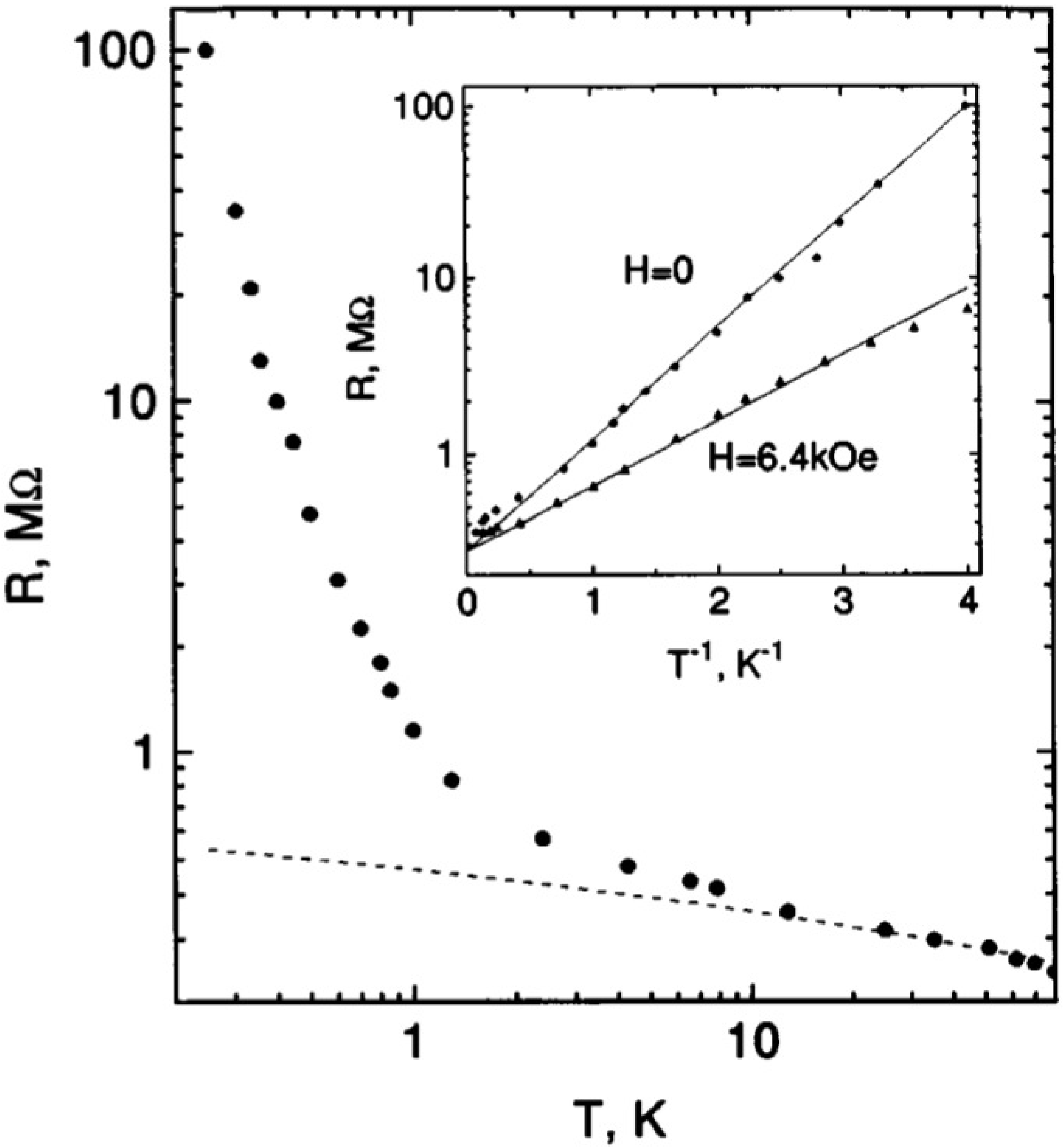
$\longrightarrow$ diffusion on σ-model curved space $\qquad \partial_t W = \Delta_Q W \; , \quad t = x/\xi$

- Localization length Efetov, Larkin '83
- Exact solution for the statistics of eigenfunctions Fyodorov, ADM '92-94
- Exact $\langle g \rangle (L/\xi)$ and $\mathrm{var}(g)(L/\xi)$ Zirnbauer, ADM, Müller-Groeling '92-94



orthogonal (full), unitary (dashed), symplectic (dot-dashed)

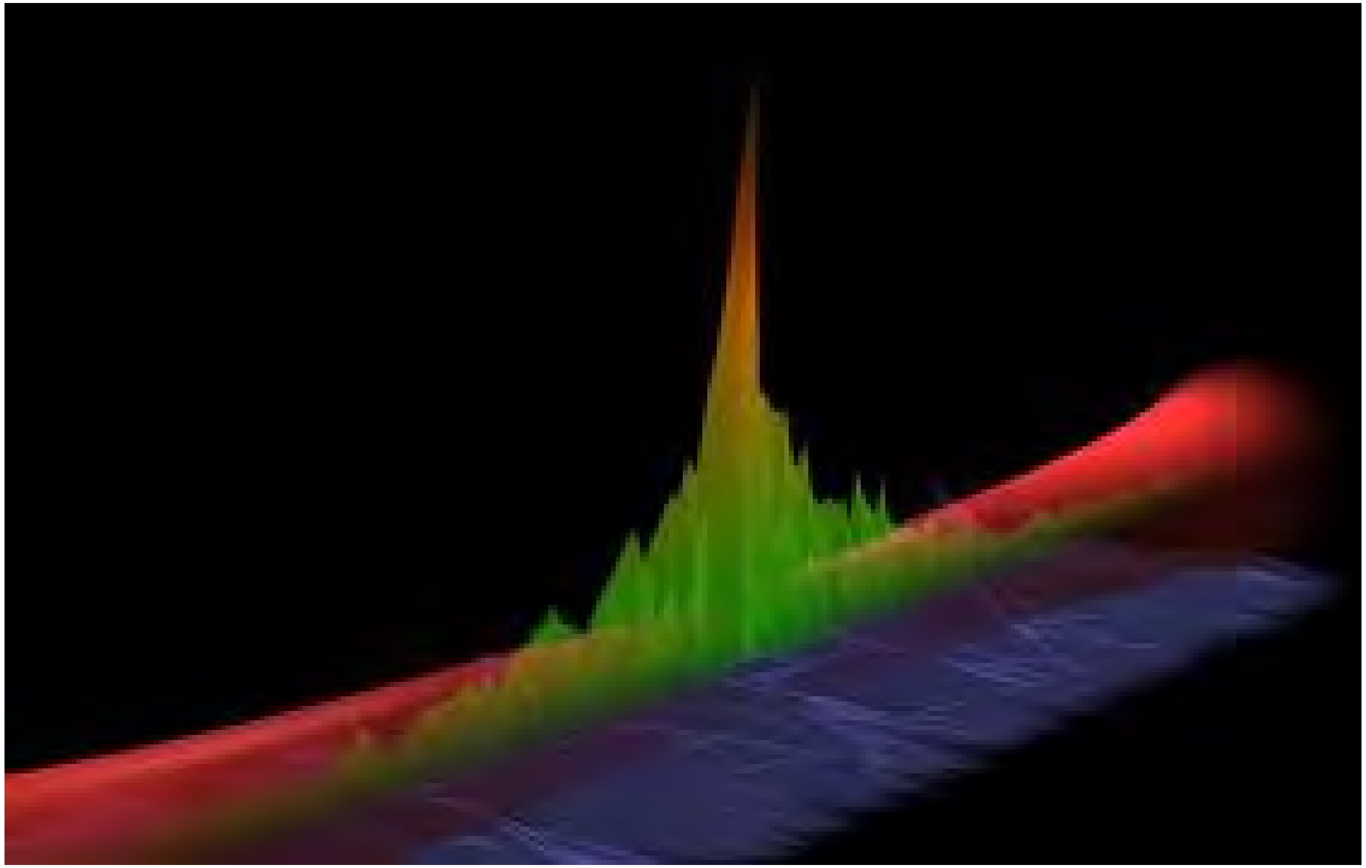
From weak to strong localization of electrons in wires



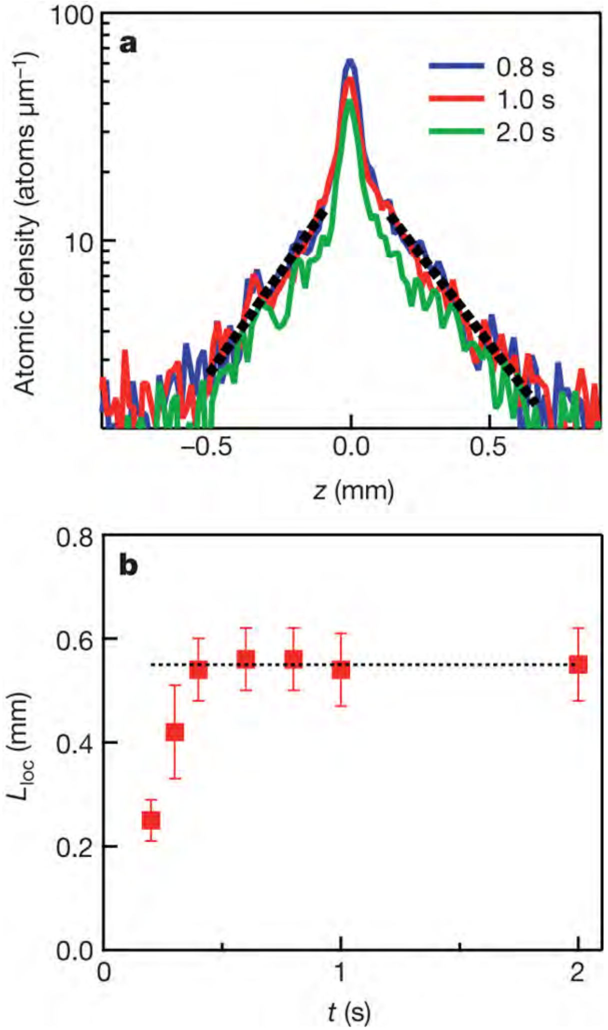
GaAs wires

Gershenson et al, PRL 97

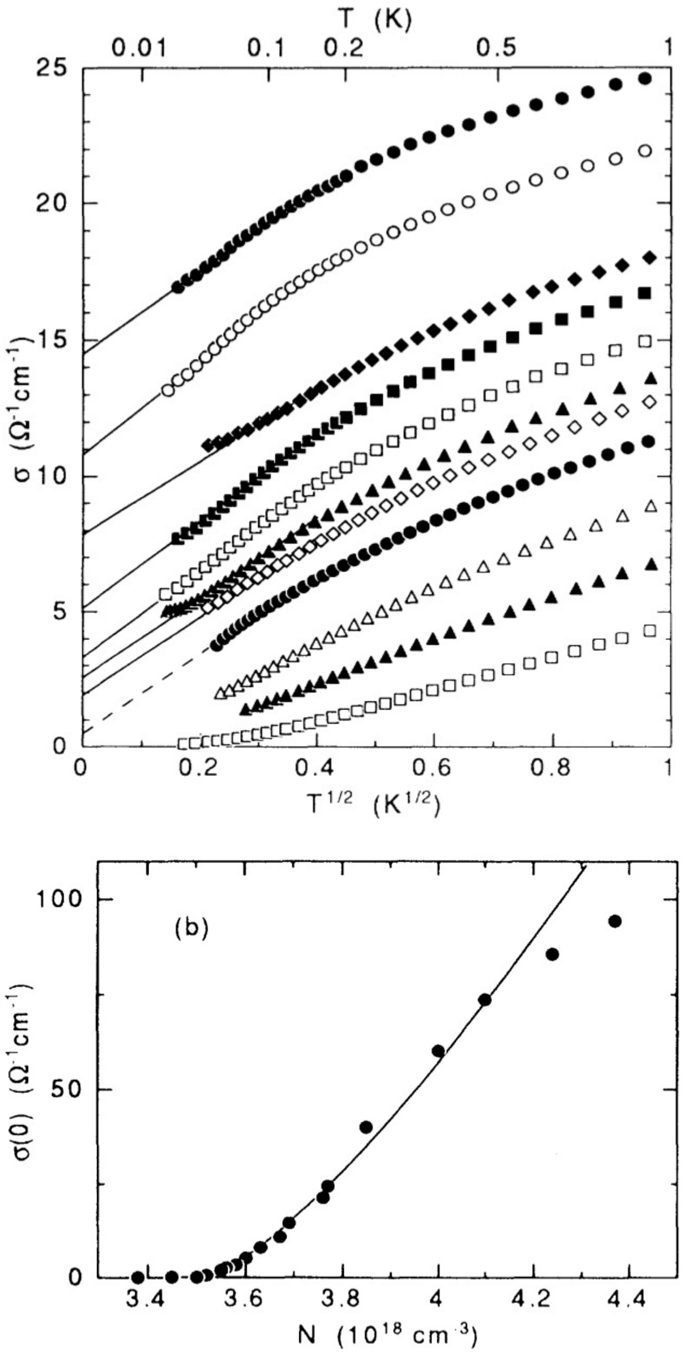
Anderson localization of atomic Bose-Einstein condensate in 1D

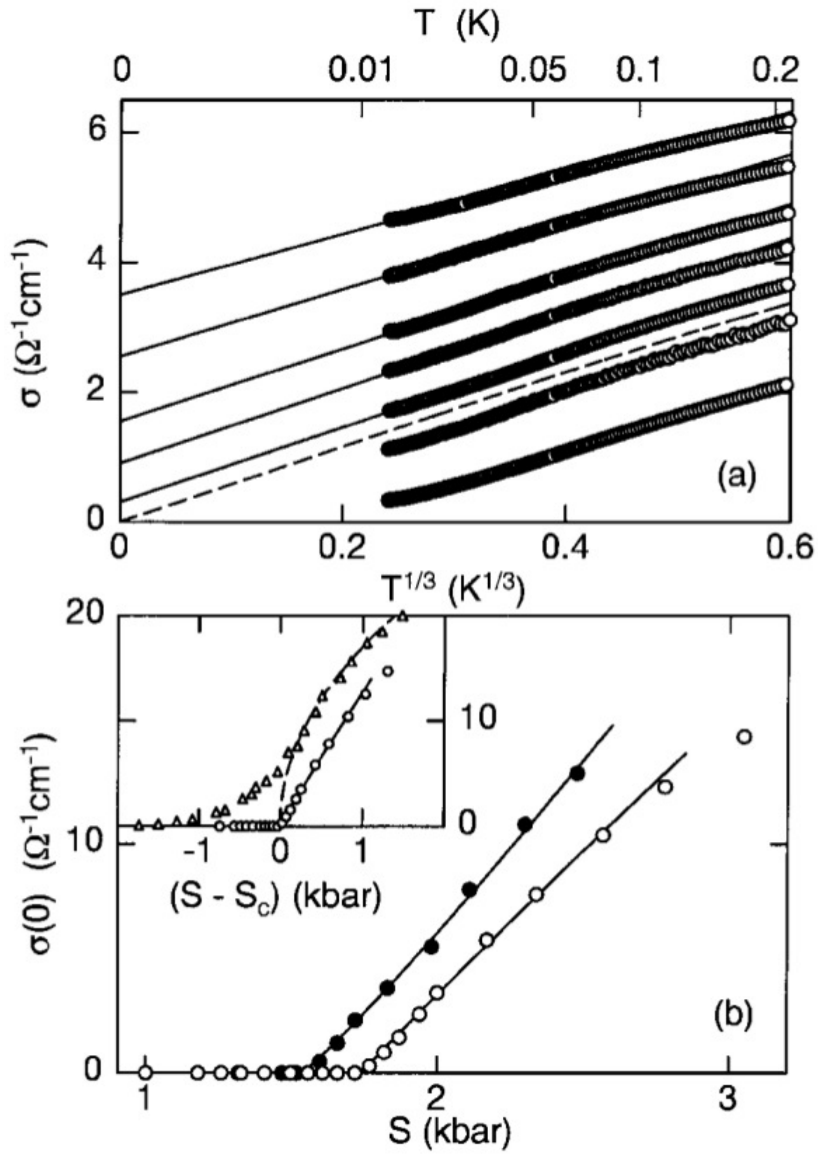


Billy et al (Aspect group), Nature 2008



3D Anderson localization transition in Si:P





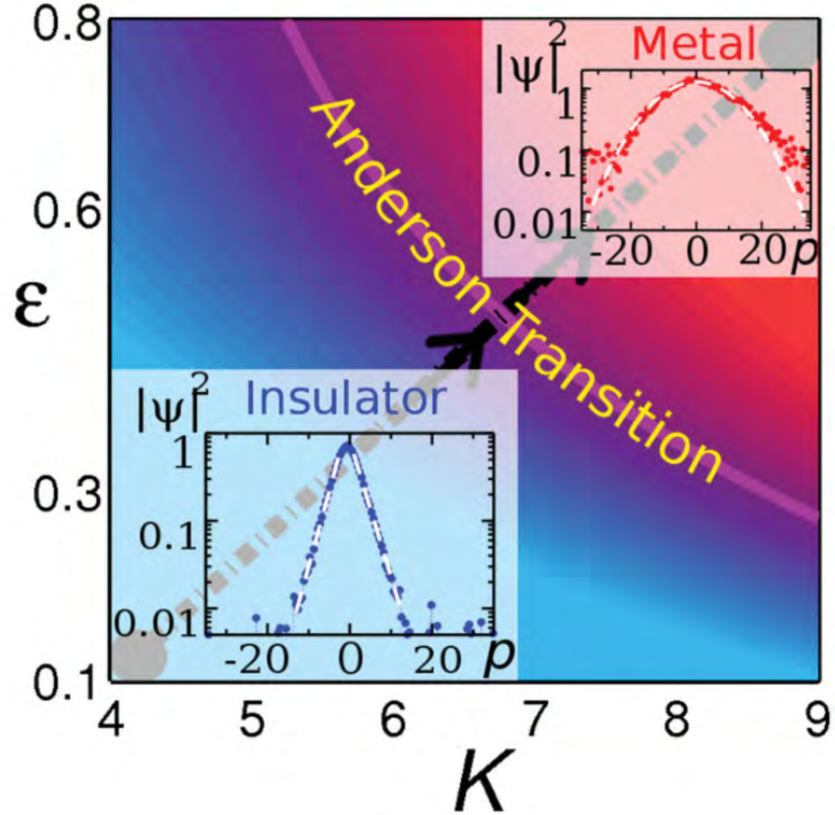
Stupp et al, PRL'93; Wafenschmidt et al, PRL'97 (von Löhneysen group)

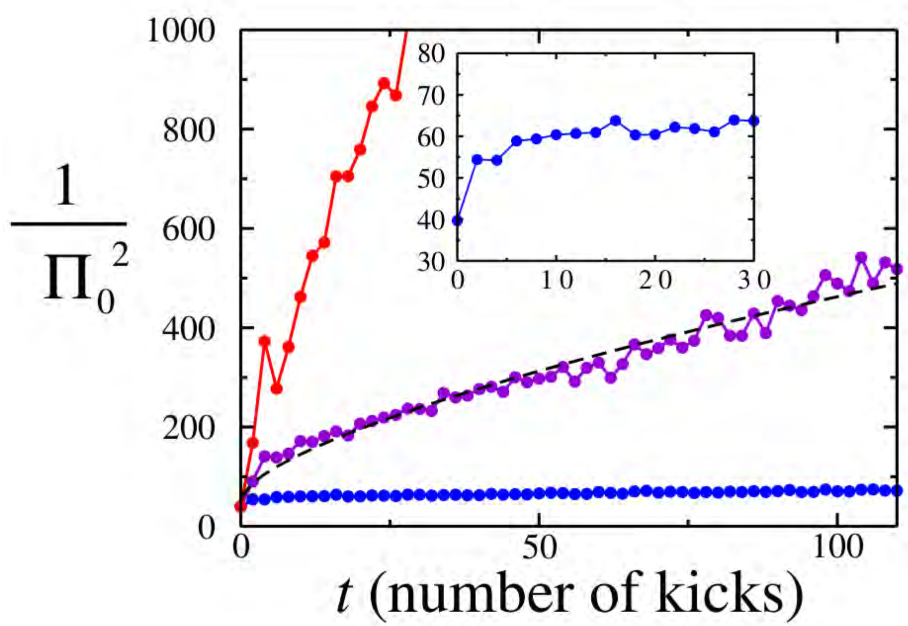
3D Anderson localization in atomic "kicked rotor"

kicked rotor $$H = \frac{p^2}{2} + K \cos x [1 + \epsilon \cos \omega_2 t \cos \omega_3 t] \sum_n \delta(t - 2\pi n/\omega_1)$$

Anderson localization in momentum space. Three frequencies mimic 3D !

Experimental realization: cesium atoms exposed to a pulsed laser beam.





Chabé et al, PRL'08

Multifractality at the Anderson transition

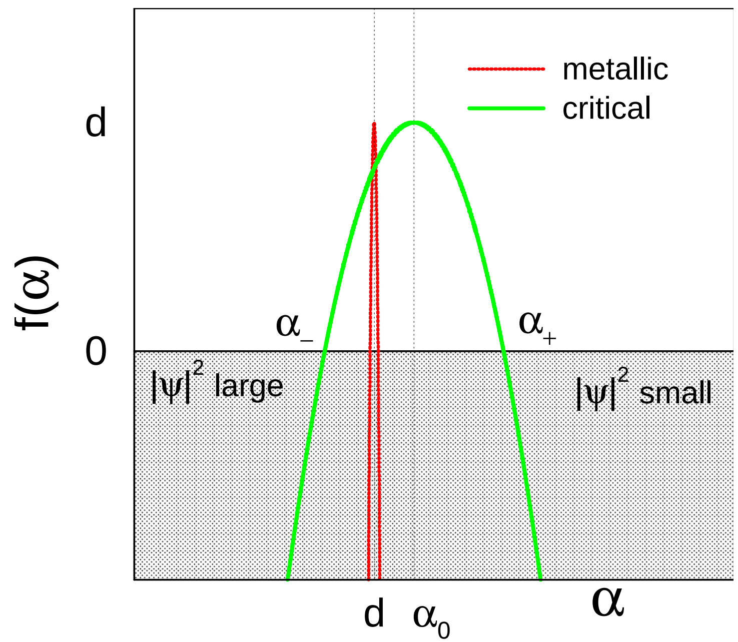
$P_q = \int d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$$\tau_q = \underbrace{d(q-1)}_{\text{normal}} + \underbrace{\Delta_q}_{\text{anomalous}} \equiv D_q(q-1) \quad \text{multifractality}$$

$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$

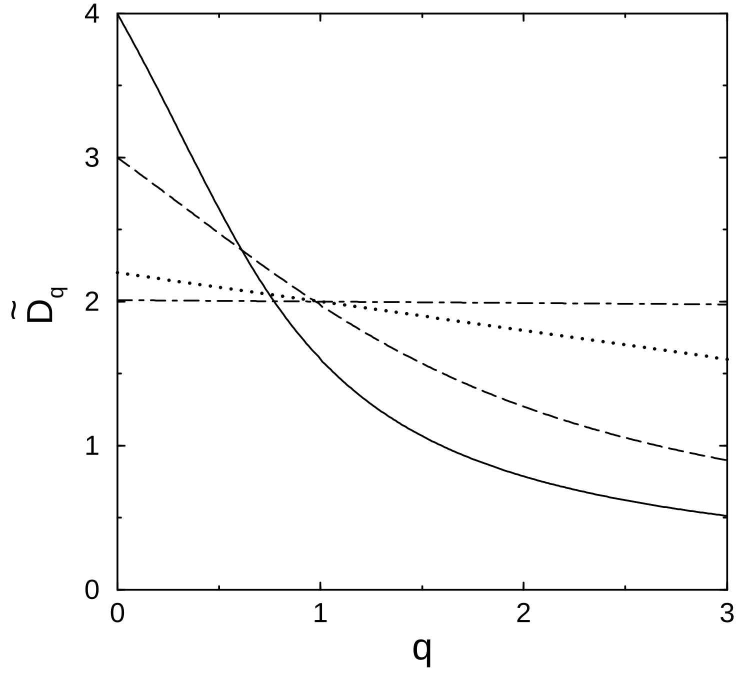


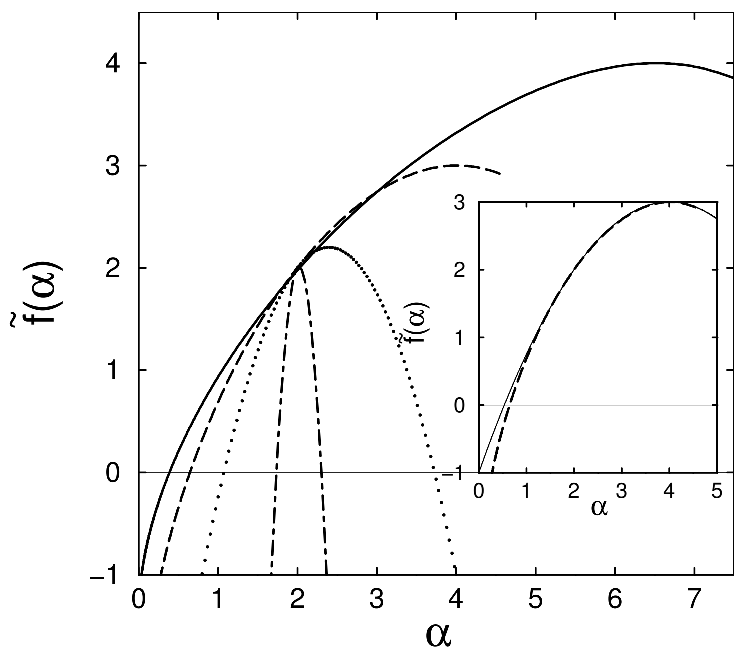
wave function statistics:

$$\mathcal{P}(\ln|\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$$

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Dimensionality dependence of multifractality





Analytics (2 + ϵ, one-loop) and numerics

$$\tau_q = (q-1)d - q(q-1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d+\epsilon-\alpha)^2/4\epsilon + O(\epsilon^4)$$

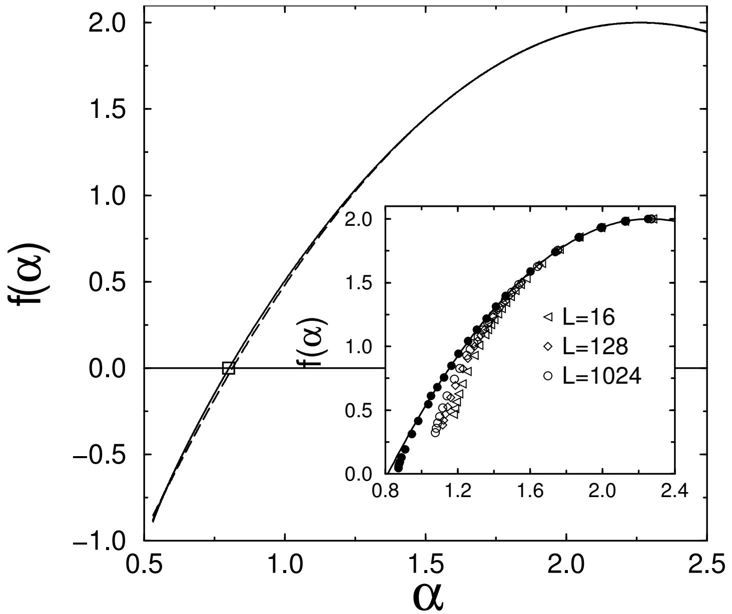
$d = 4$ (full)
$d = 3$ (dashed)
$d = 2 + \epsilon, \quad \epsilon = 0.2$ (dotted)
$d = 2 + \epsilon, \quad \epsilon = 0.01$ (dot-dashed)

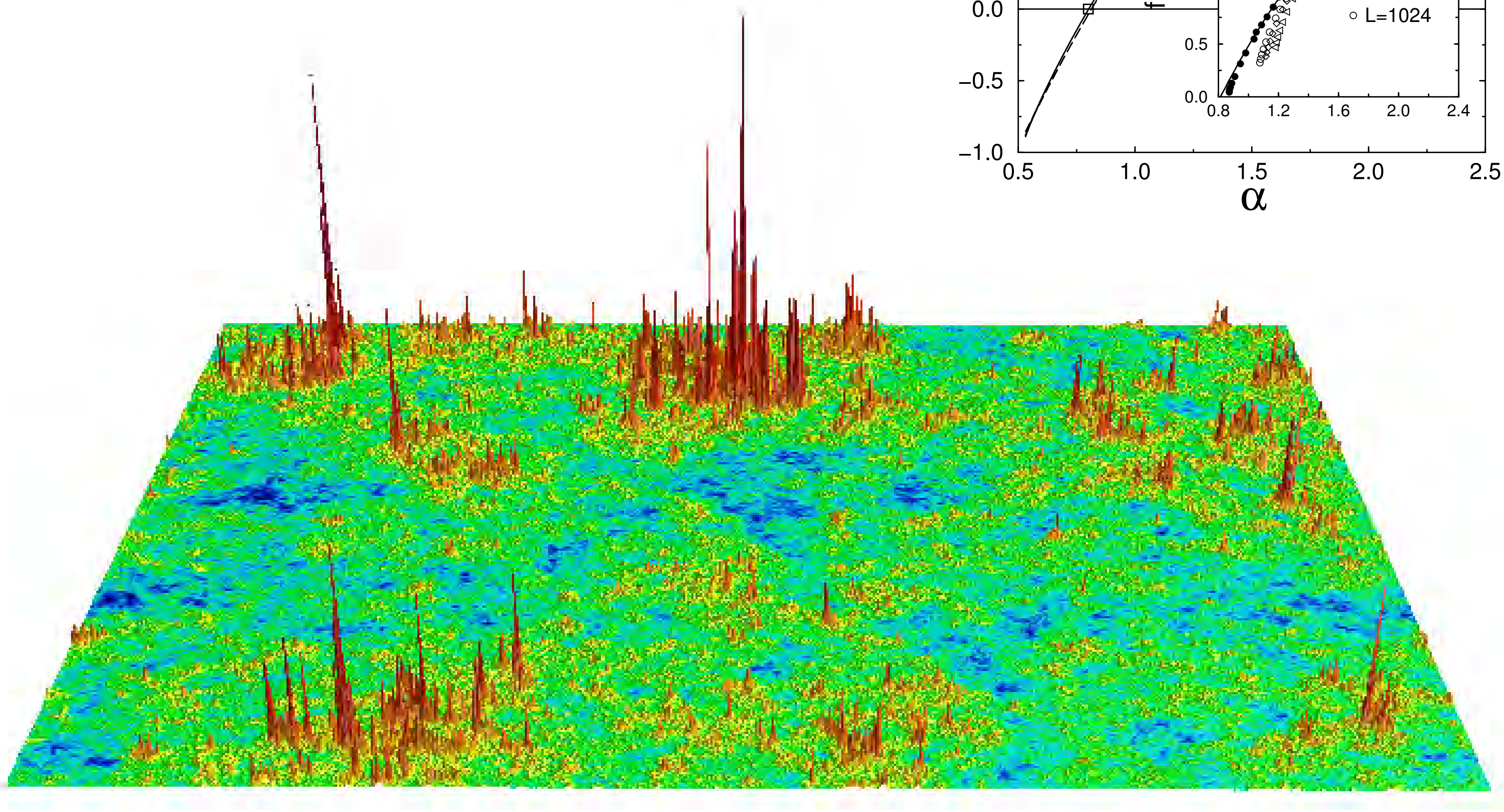
Inset: $d = 3$ (dashed)
vs. $d = 2 + \epsilon, \quad \epsilon = 1$ (full)

Mildenberger, Evers, ADM '02

Multifractality at the Quantum Hall transition

Evers, Mildenberger, ADM '01





Power-law random banded matrix model (PRBM)

ADM, Fyodorov, Dittes, Quezada, Seligman '96

$N \times N$ random matrix $H = H^\dagger$ $\qquad \langle |H_{ij}|^2 \rangle = \dfrac{1}{1 + |i-j|^2/b^2}$

$\longleftrightarrow$ 1D model with $1/r$ long range hopping

$0 < b < \infty$ parameter

Critical for any b $\longrightarrow$ family of critical theories!

$b \gg 1$ analogous to $d = 2 + \epsilon$ $\qquad$ $b \ll 1$ analogous to $d \gg 1$

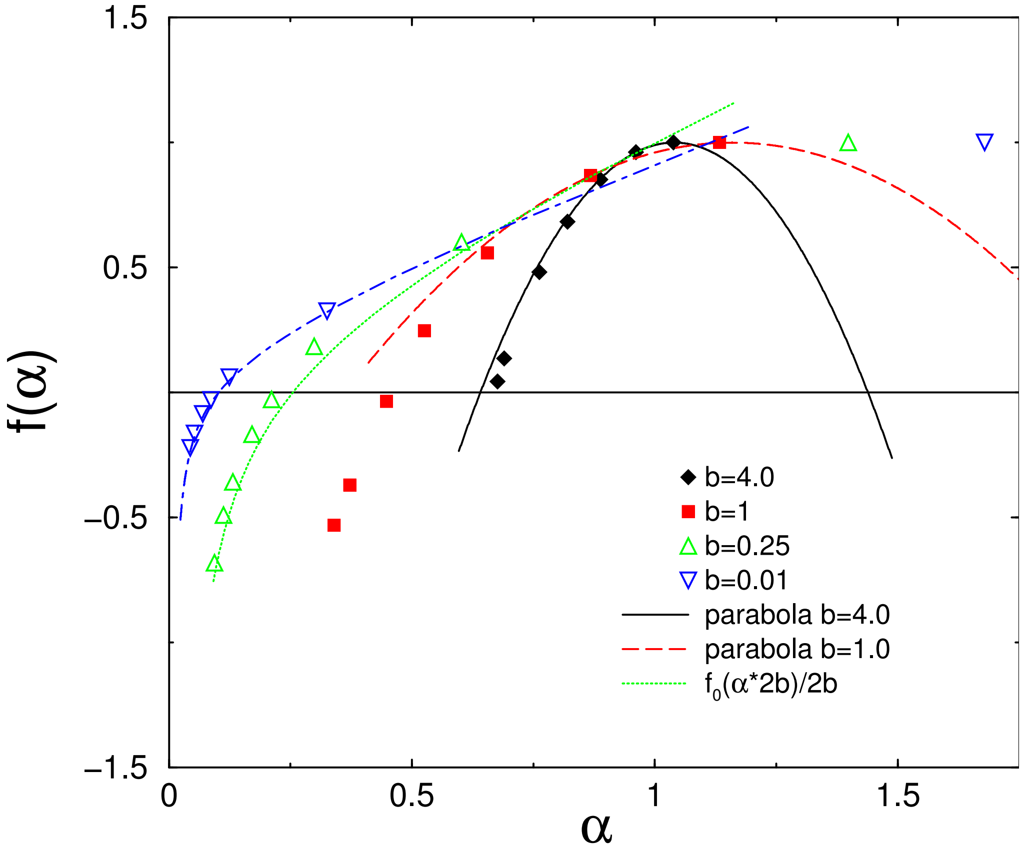
Analytics: $b \gg 1$: σ-model RG

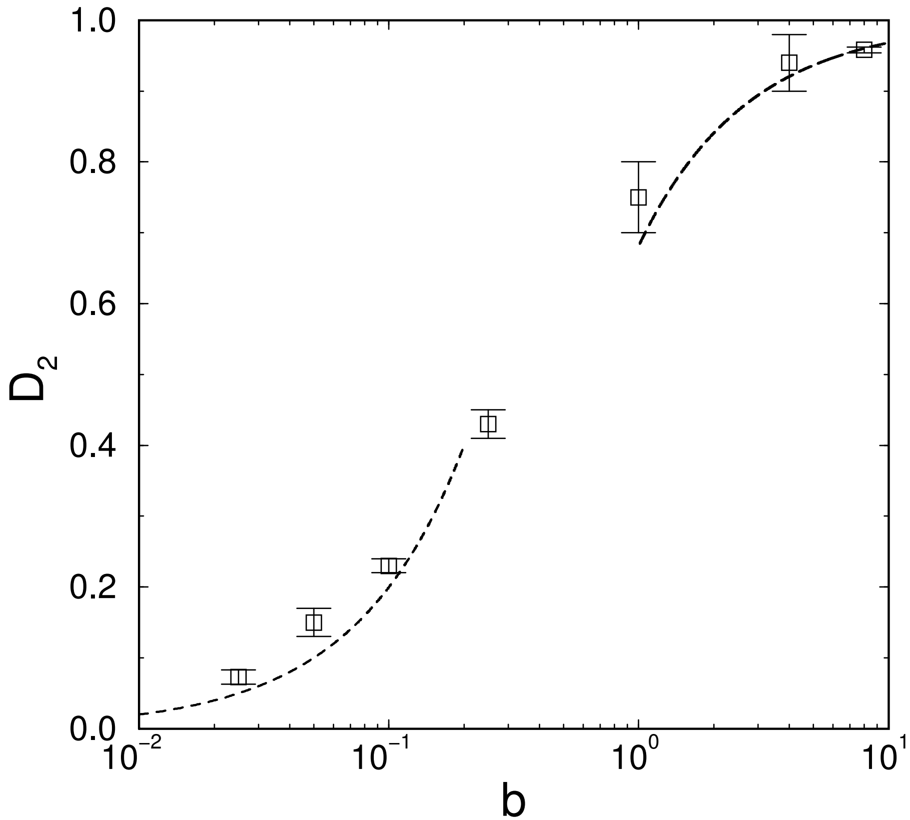
$b \ll 1$: real space RG

Numerics: efficient in a broad range of b

Evers, ADM '01

Multifractality in PRBM model: analytics vs numerics





numerics: $b = 4, \; 1, \; 0.25, \; 0.01$

analytics: $b \gg 1$ (σ–model RG), $b \ll 1$ (real-space RG)

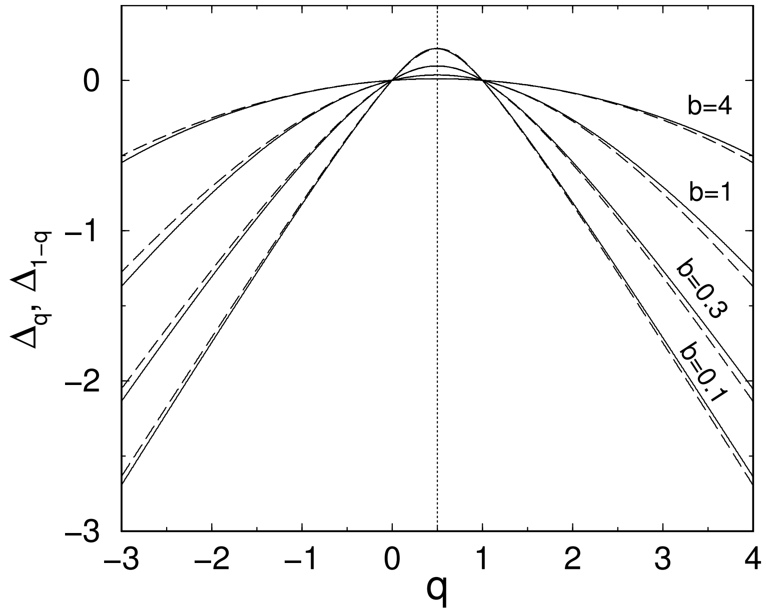
Symmetry of multifractal spectra

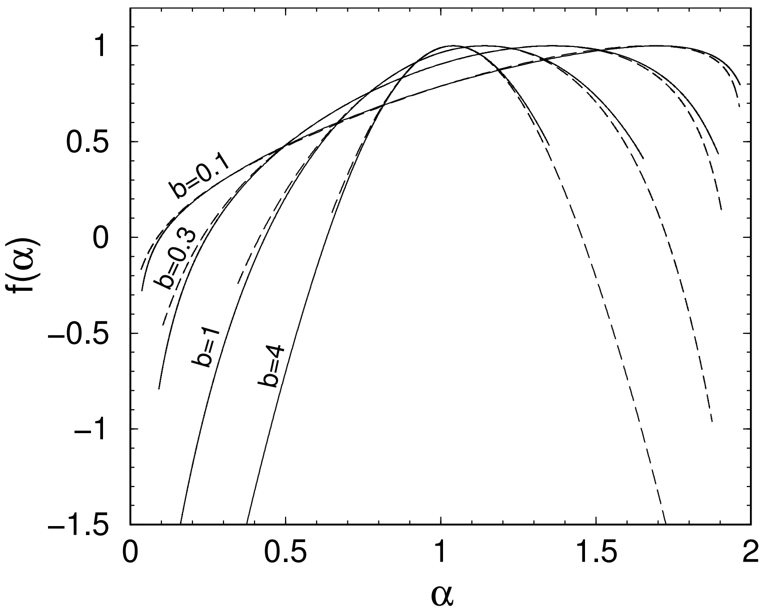
ADM, Fyodorov, Mildenberger, Evers '06

LDOS distribution in σ-model + **universality**

⟶ **exact symmetry of the multifractal spectrum:**

$$\Delta_q = \Delta_{1-q} \qquad\qquad f(2d-\alpha) = f(\alpha) + d - \alpha$$





⟶ **probabilities of unusually large and unusually small $|\psi^2(r)|$ are related !**

Multifractality: Generalizations

- Symmetry of multifractal spectra as a consequence of invariance of the σ model correlation functions with respect to Weyl group of the σ model target space;

generalization to unconventional symmetry classes

Gruzberg, Ludwig, ADM, Zirnbauer PRL'11

- generalization on full set of composite operators,

i.e. also on subleading ones.

Gruzberg, ADM, Zirnbauer, PRB'13

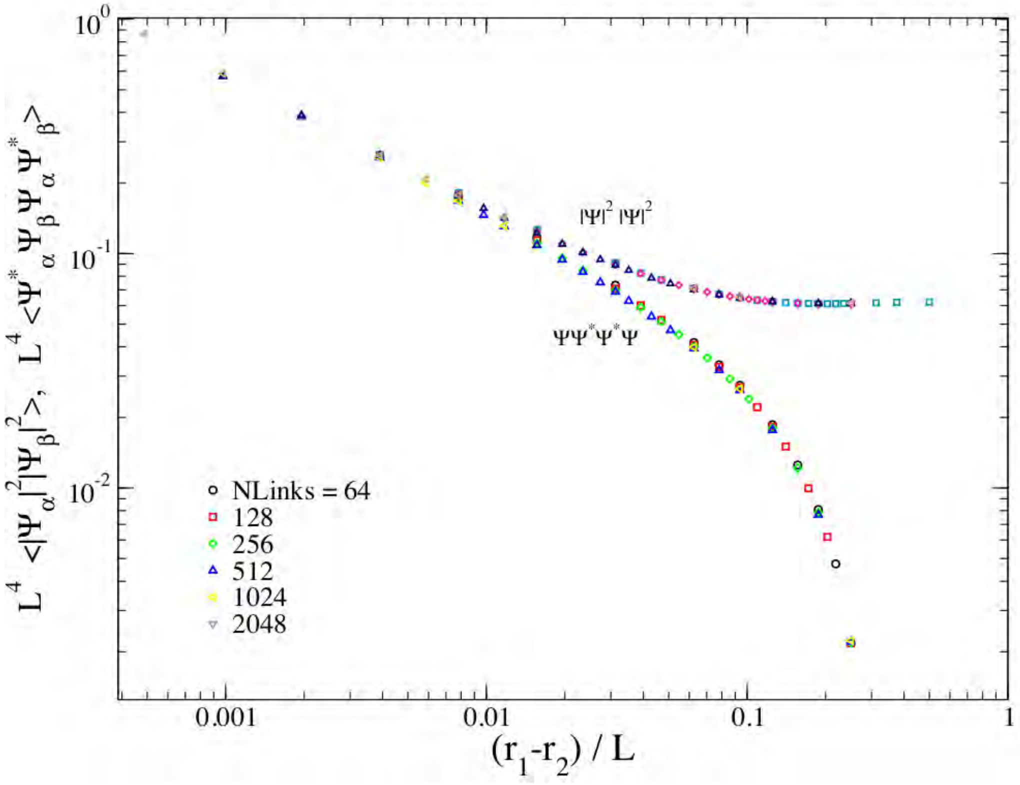
Important example:

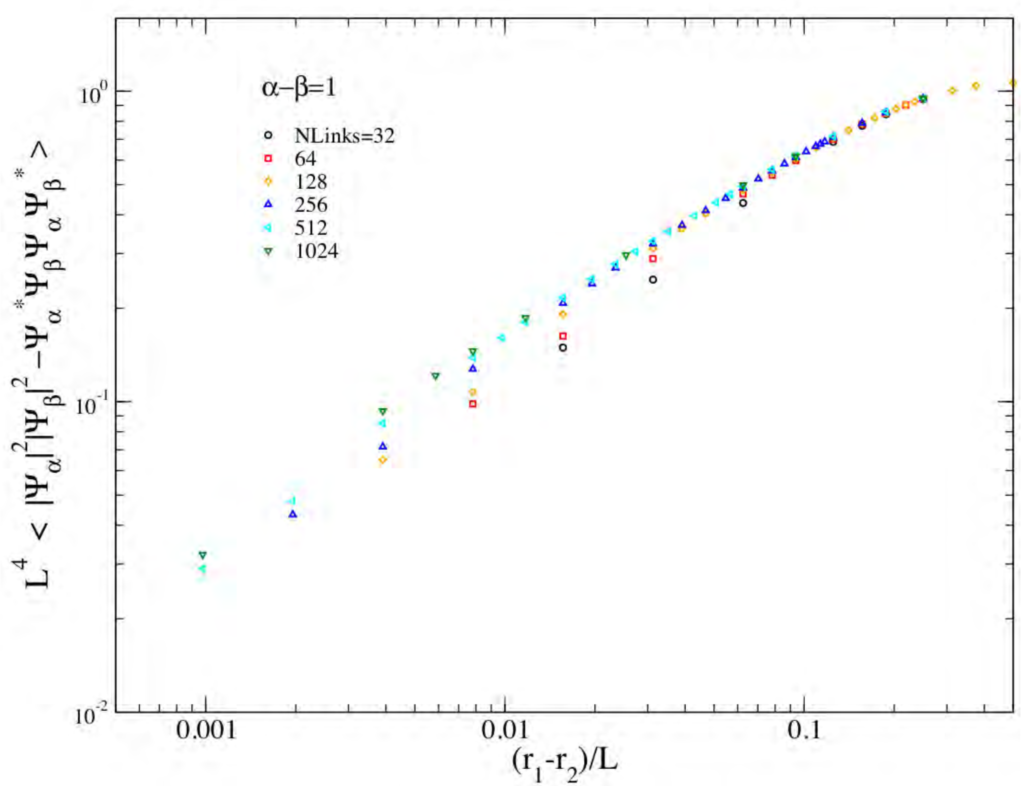
$$A_2 = V^2 |\psi_1(r_1)\psi_2(r_2) - \psi_1(r_2)\psi_2(r_1)|^2$$

$\longleftrightarrow$ Hartree-Fock matrix element of e-e interaction

scaling: $\langle A_2^q \rangle \propto L^{-\Delta_q^{(2)}}$ symmetry: $\Delta_q^{(2)} = \Delta_{2-q}^{(2)}$

Interaction scaling at criticality





Hartree, Fock

enhanced by multifractality

exponent $\Delta_2 \simeq -0.52 < 0$

Hartree – Fock

suppressed by multifractality

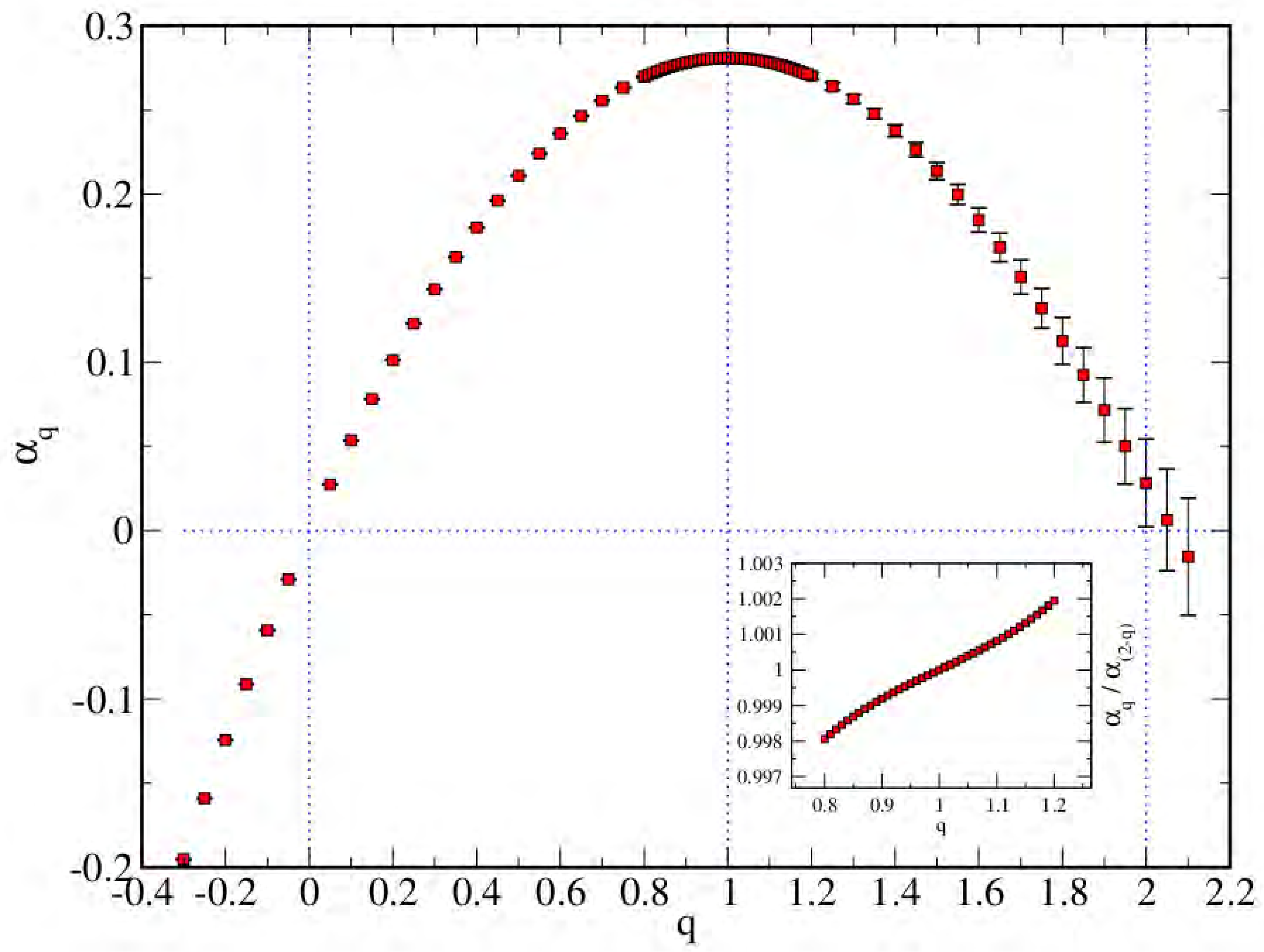
exponent $\Delta_1^{(2)} \simeq 0.62 > 0$

Burmistrov, Bera, Evers, Gornyi, ADM, Annals Phys. 326, 1457 (2011)

$\longrightarrow$ Temperature scaling at quantum Hall and metal-insulator transitions with short-range interaction

Multifractal spectrum of A_2 at quantum Hall transition

Numerical data: Bera, Evers, unpublished

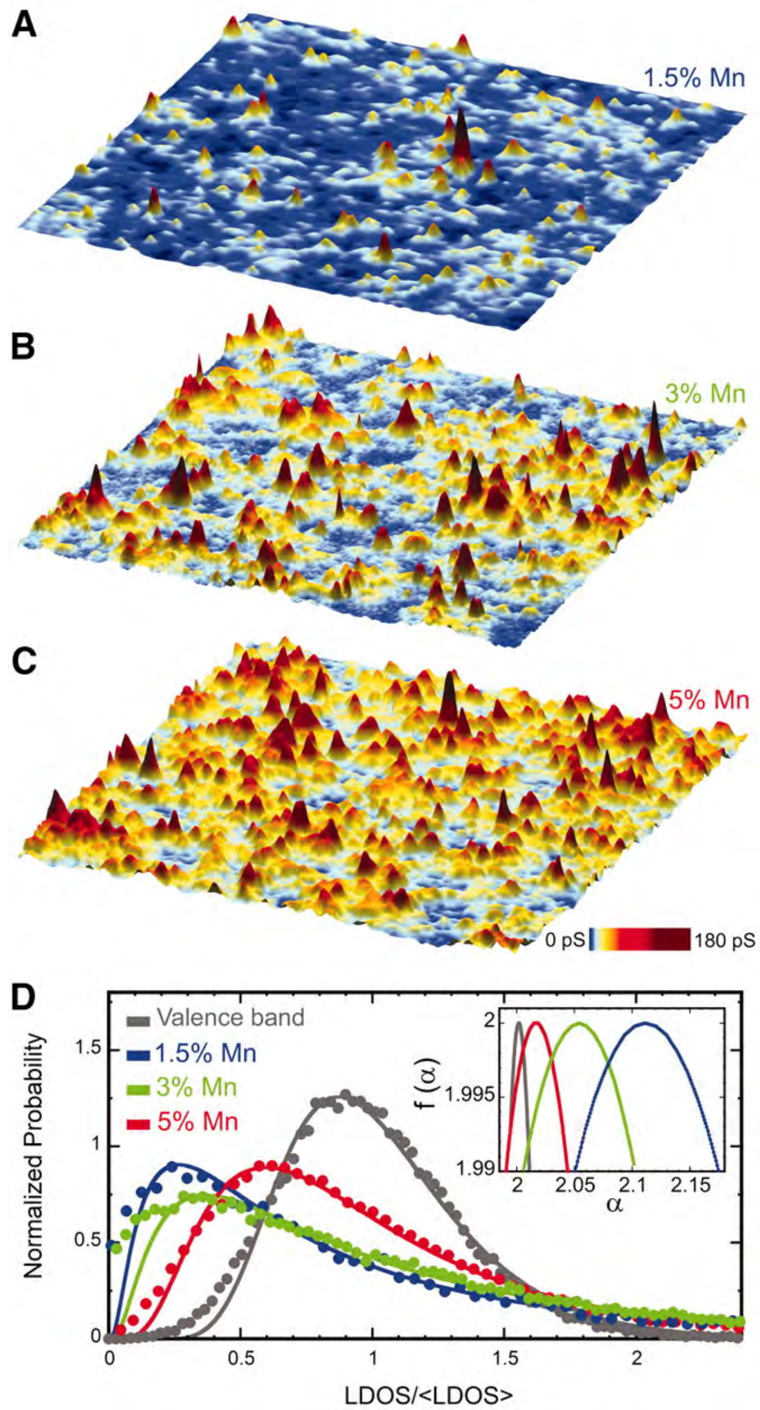


Confirms the symmetry $\quad q \longleftrightarrow 2 - q$

Multifractality: Experiment I

Local DOS flucutuations near metal-insulator transition in $Ga_{1-x}Mn_xAs$

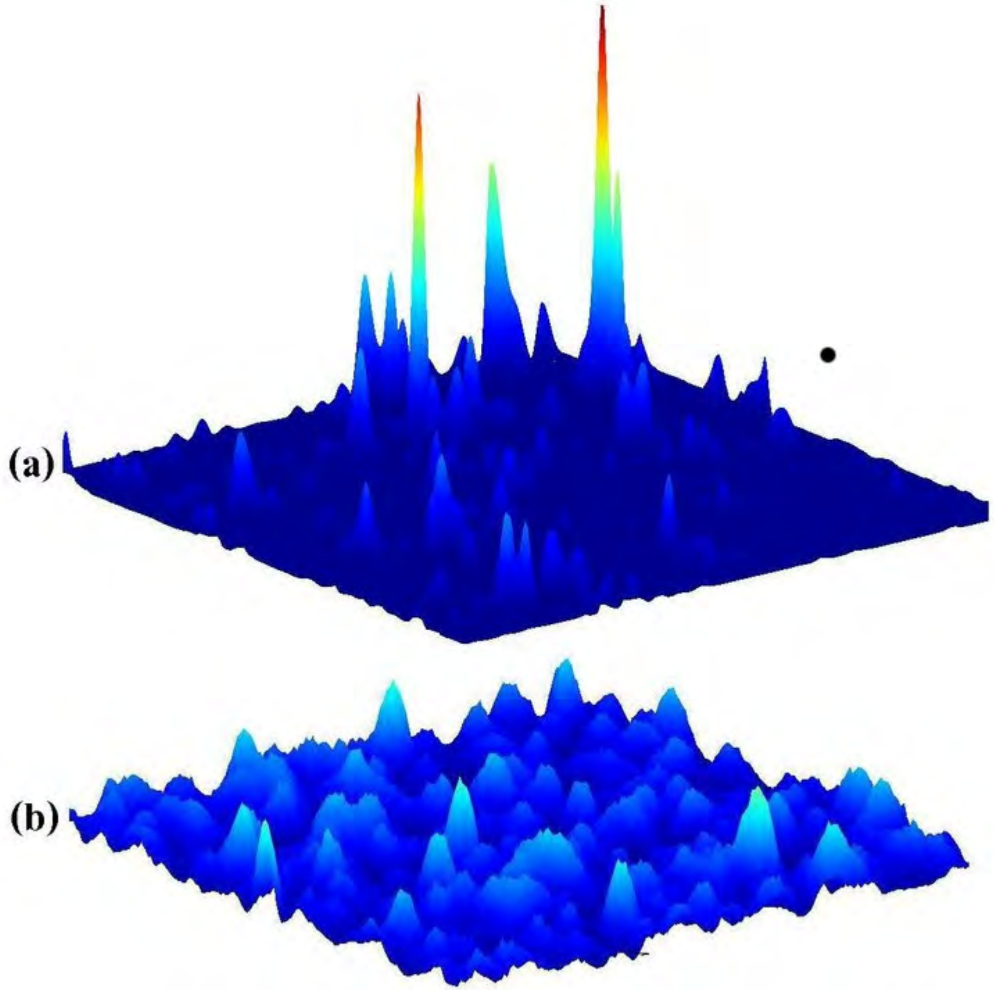
Richardella,...,Yazdani, Science '10

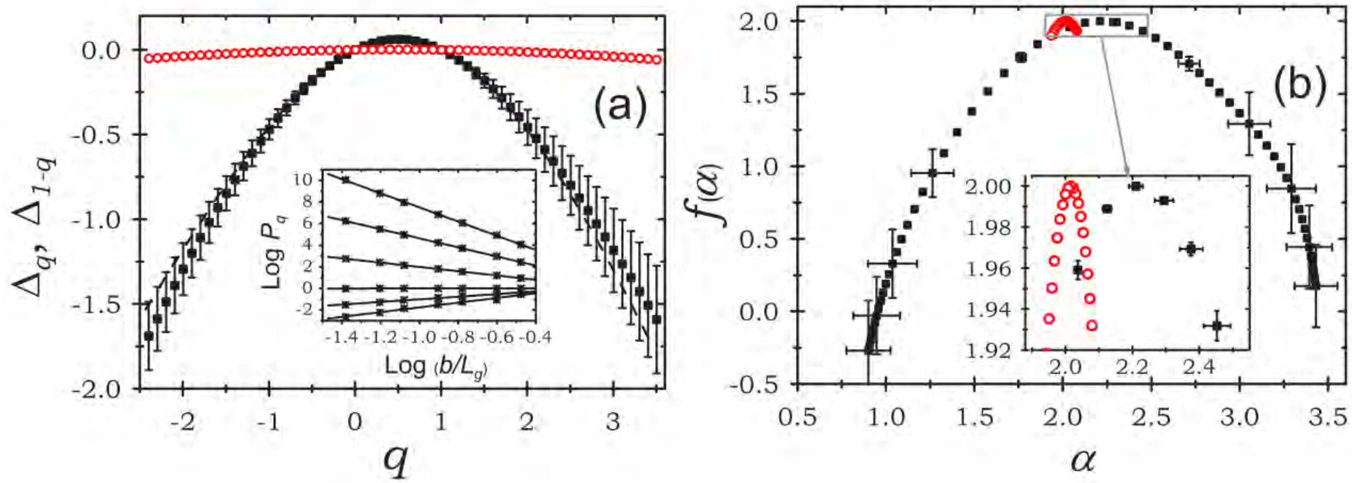


Multifractality: Experiment II

Ultrasound speckle in a system of randomly packed Al beads

Faez, Strybulevich, Page, Lagendijk, van Tiggelen, PRL'09

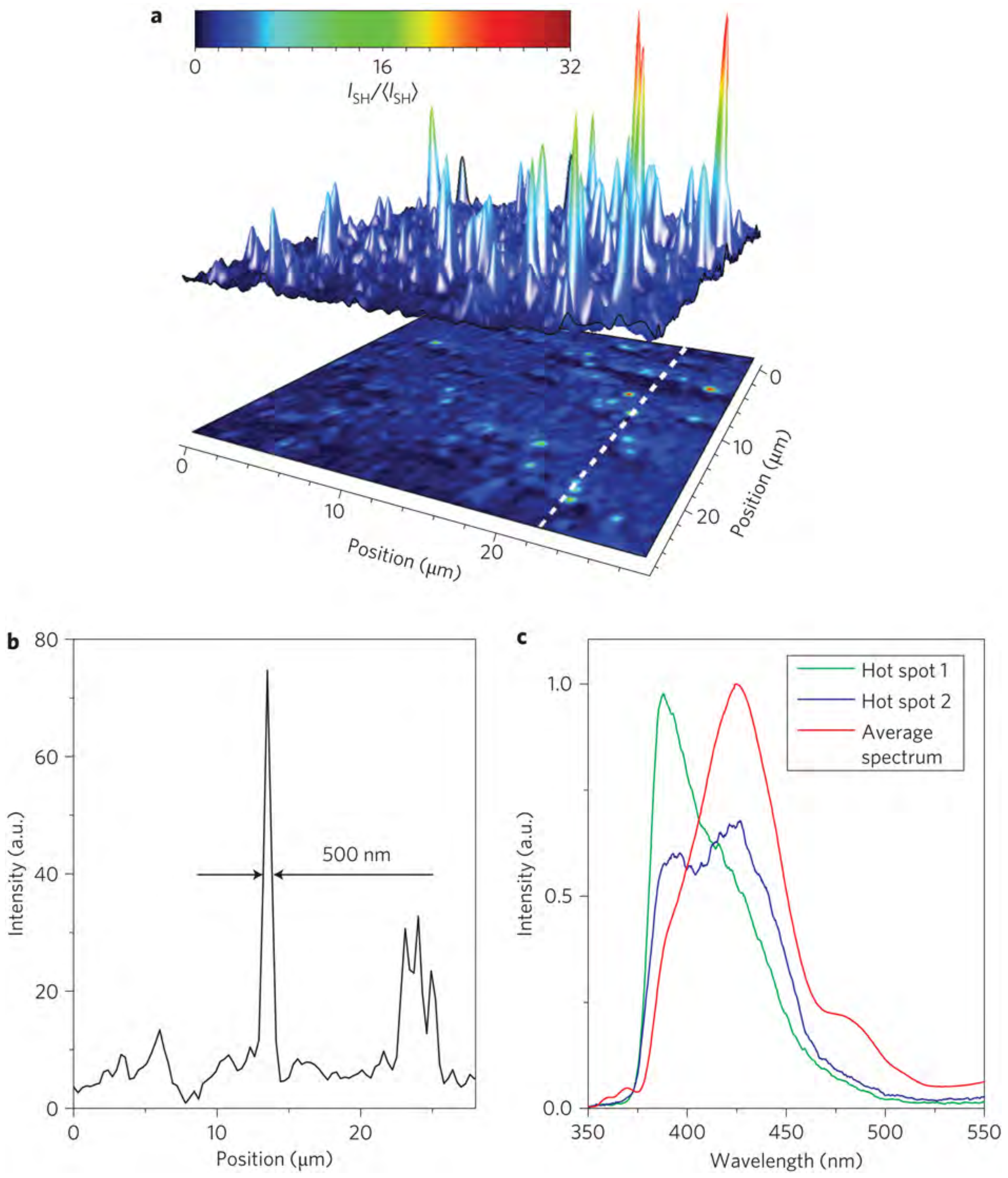




Multifractality: Experiment III

Localization of light in an array of dielectric nano-needles

Mascheck et al, Nature Photonics '12



Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	symbol
GOE	+	+	AI
GUE	−	+/−	A
GSE	+	−	AII

Chiral classes

	T	spin rot.	symbol
ChOE	+	+	BDI
ChUE	−	+/−	AIII
ChSE	+	−	CII

$$H = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}$$

Bogoliubov-de Gennes classes

	T	spin rot.	symbol
	+	+	CI
	−	+	C
	+	−	DIII
	−	−	D

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^T \end{pmatrix}$$

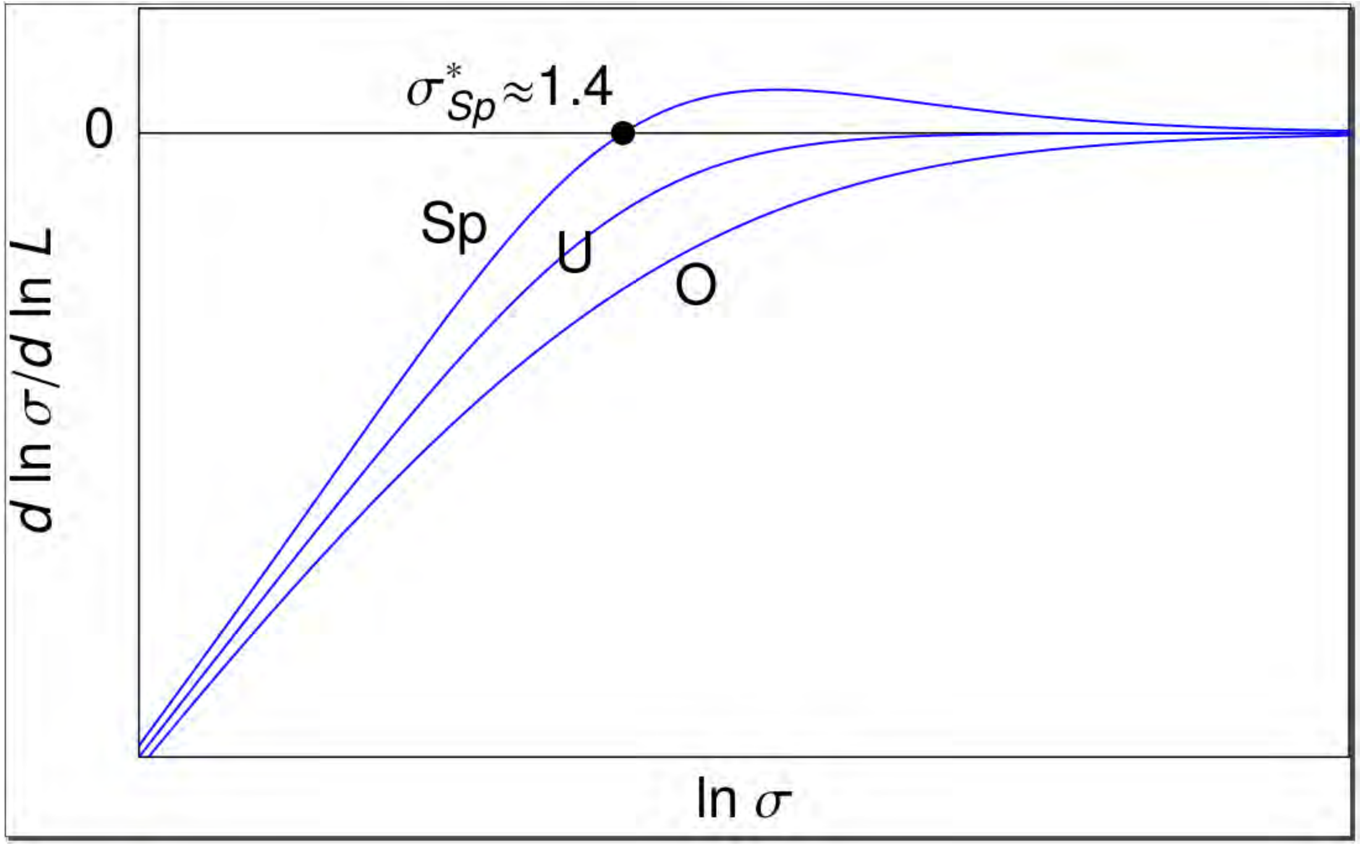
Disordered electronic systems: Symmetry classification

$\longleftrightarrow$ classification of symmetric spaces Zirnbauer'96, Altland, Zirnbauer'97

Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$U(N)$	$GL(N,\mathbb{C})/U(N)$	AIII|AIII	$U(2n)/U(n)\times U(n)$
AI	GOE	$+$	$+$	$U(N)/O(N)$	$GL(N,\mathbb{R})/O(N)$	BDI|CII	$Sp(4n)/Sp(2n)\times Sp(2n)$
AII	GSE	$+$	$-$	$U(2N)/Sp(2N)$	$U^*(2N)/Sp(2N)$	CII|BDI	$O(2n)/O(n)\times O(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$U(p+q)/U(p)\times U(q)$	$U(p,q)/U(p)\times U(q)$	A|A	$U(n)$
BDI	chGOE	$+$	$+$	$SO(p+q)/SO(p)\times SO(q)$	$SO(p,q)/SO(p)\times SO(q)$	AI|AII	$U(2n)/Sp(2n)$
CII	chGSE	$+$	$-$	$Sp(2p+2q)/Sp(2p)\times Sp(2q)$	$Sp(2p,2q)/Sp(2p)\times Sp(2q)$	AII|AI	$U(n)/O(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$Sp(2N)$	$Sp(2N,\mathbb{C})/Sp(2N)$	DIII|CI	$Sp(2n)/U(n)$
CI		$+$	$+$	$Sp(2N)/U(N)$	$Sp(2N,\mathbb{R})/U(N)$	D|C	$Sp(2n)$
BD		$-$	$-$	$SO(N)$	$SO(N,\mathbb{C})/SO(N)$	CI|DIII	$O(2n)/U(n)$
DIII		$+$	$-$	$SO(2N)/U(N)$	$SO^*(2N)/U(N)$	C|D	$O(n)$

Universality classes: Spatial dimenionality, symmetry, topology

Role of symmetry: 2D systems of Wigner-Dyson classes

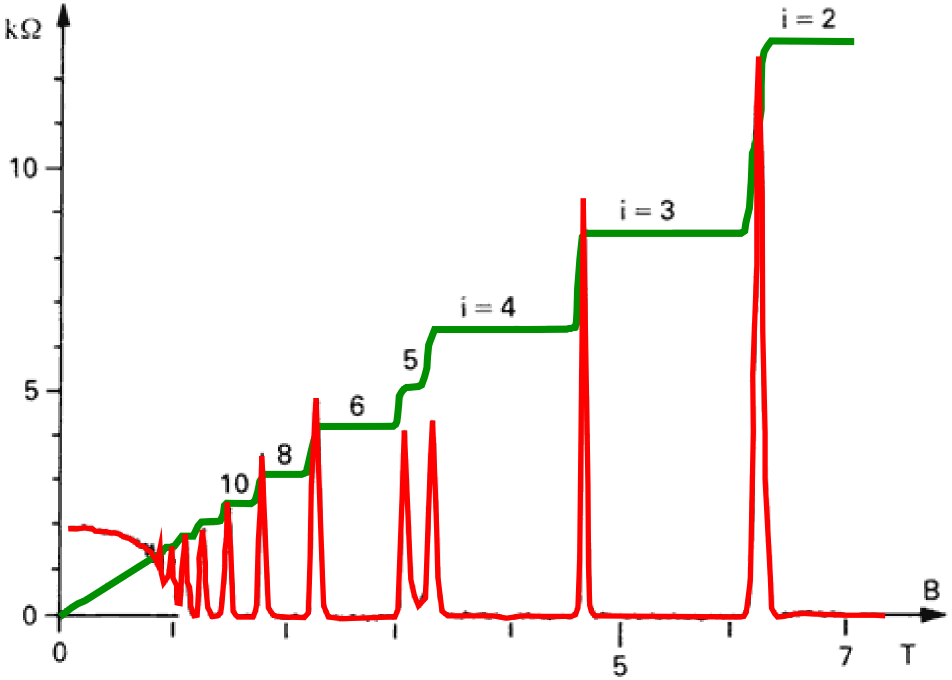


Orthogonal and Unitary: localization;
parametrically different localization length: $\xi_U \gg \xi_O$

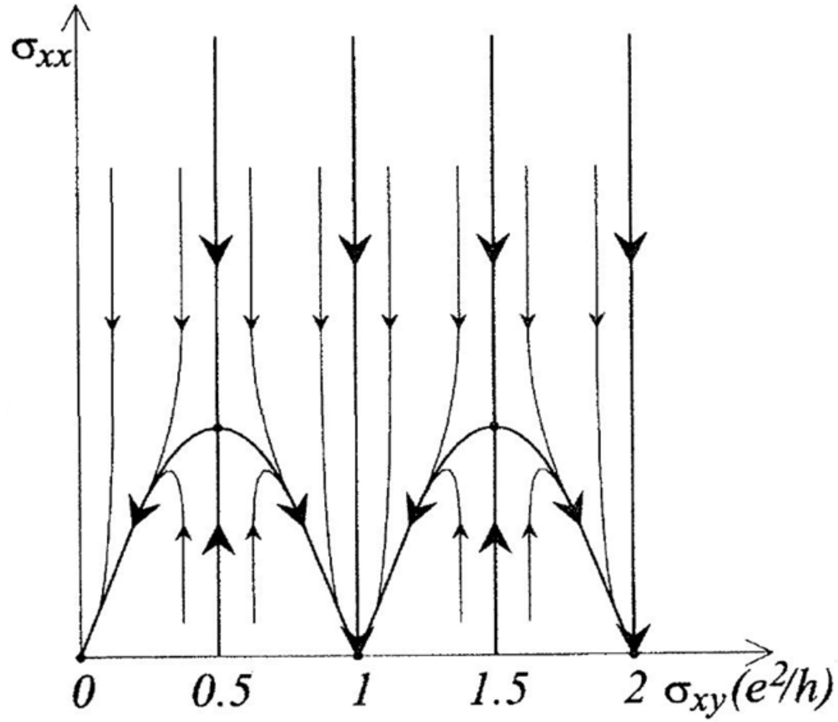
Symplectic: metal-insulator transition

Usual realization of Sp class: spin-orbit interaction

Anderson localization & topology: Integer Quantum Hall Effect

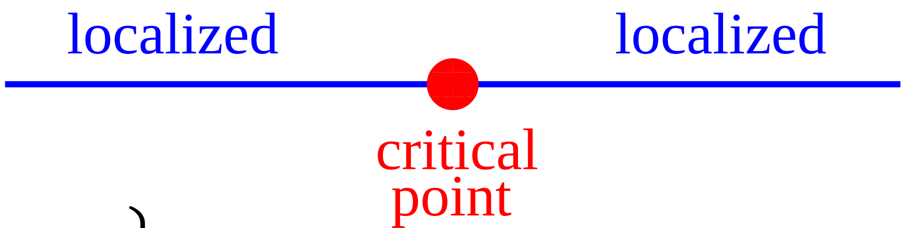


von Klitzing ’80 ; Nobel Prize ’85



IQHE flow diagram

Khmelnitskii’ 83, Pruisken’ 84



Field theory (Pruisken):

σ-model with topological term

$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr}\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

QH insulators $\longrightarrow$ $n = \ldots, -2, -1, 0, 1, 2, \ldots$ **protected edge states**

$\longrightarrow$ $\mathbb{Z}$ **topological insulator**

Periodic table of Topological Insulators

	Symmetry classes			Topological insulators				
p	H_p	R_p	S_p	$\pi_0(R_p)$	d=1	d=2	d=3	d=4
0	AI	BDI	CII	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
1	BDI	BD	AII	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
2	BD	DIII	DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
3	DIII	AII	BD	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
4	AII	CII	BDI	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
5	CII	C	AI	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	C	CI	CI	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
7	CI	AI	C	0	0	0	$\mathbb{Z}$	0
$0'$	A	AIII	AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$1'$	AIII	A	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

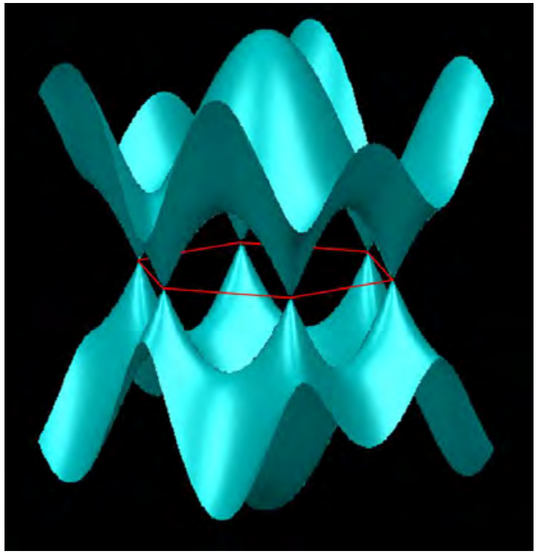
H_p – symmetry class of Hamiltonians

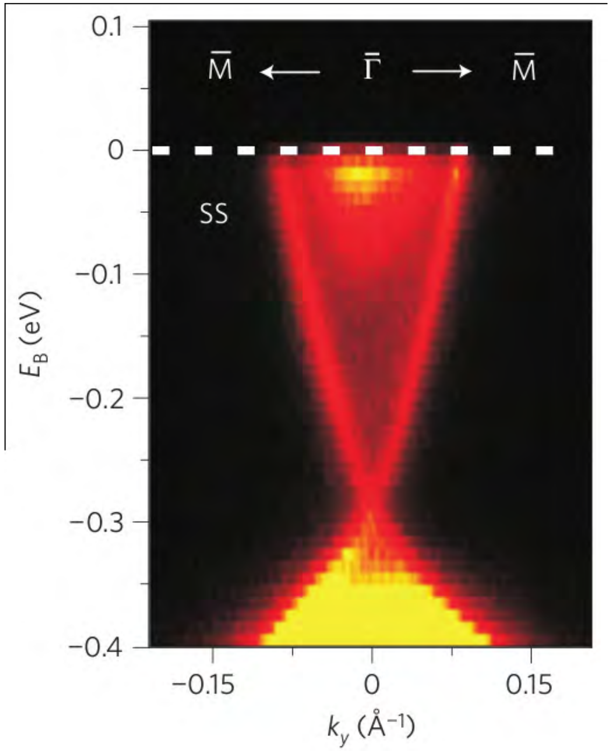
R_p – sym. class of classifying space (of Hamiltonians with eigenvalues $\to \pm 1$)

S_p – symmetry class of compact sector of σ-model manifold

Kitaev'09; Schnyder, Ryu, Furusaki, Ludwig'09; Ostrovsky, Gornyi, ADM'10

2D massless Dirac fermions





Graphene

Geim, Novoselov'04

Nobel Prize'10

Surface of 3D topological insulators

BiSb, BiSe, BiTe Hasan group '08

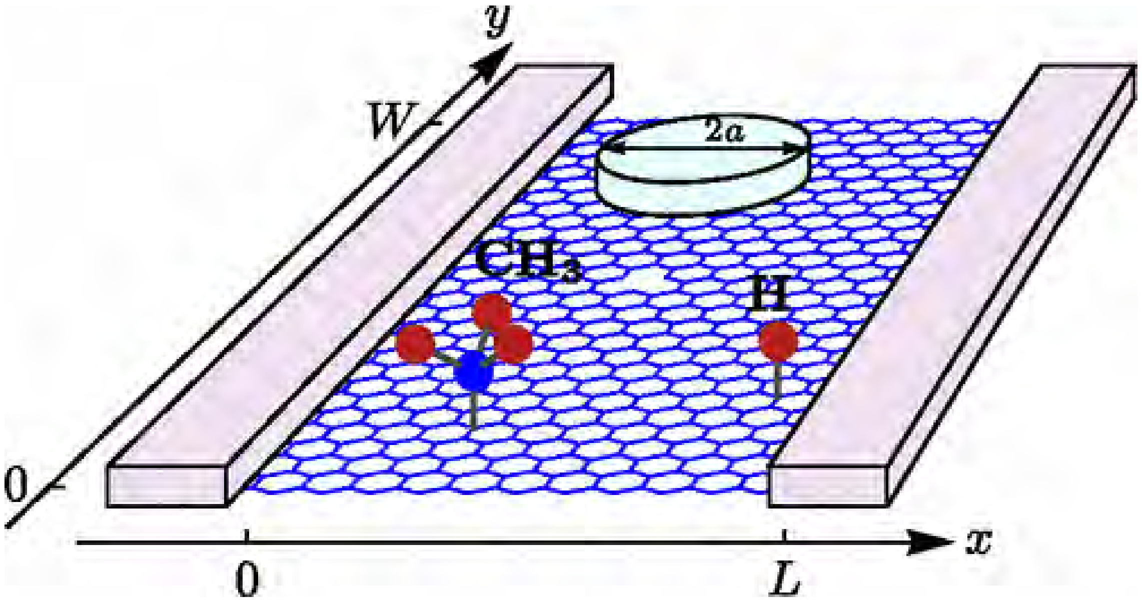
σ-model field theory with a topological term

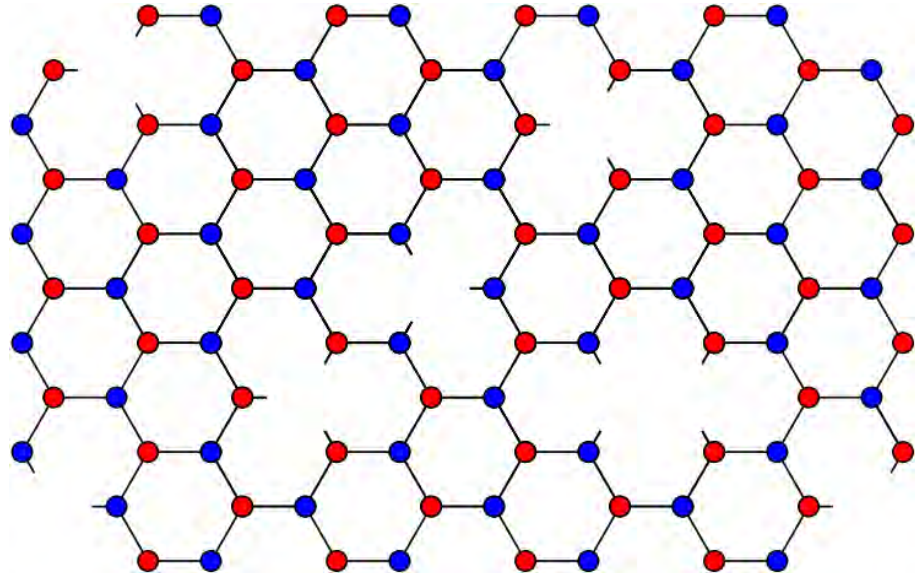
Ostrovsky, Gornyi, ADM '07

- Graphene: long-range disorder (no valley mixing)
- Surface states of 3D TI: no restriction on disorder range

Role of symmetry and topology: Graphene at the Dirac point

Ostrovsky et al, PRL'10; Gattenlöhner et al, PRL'14



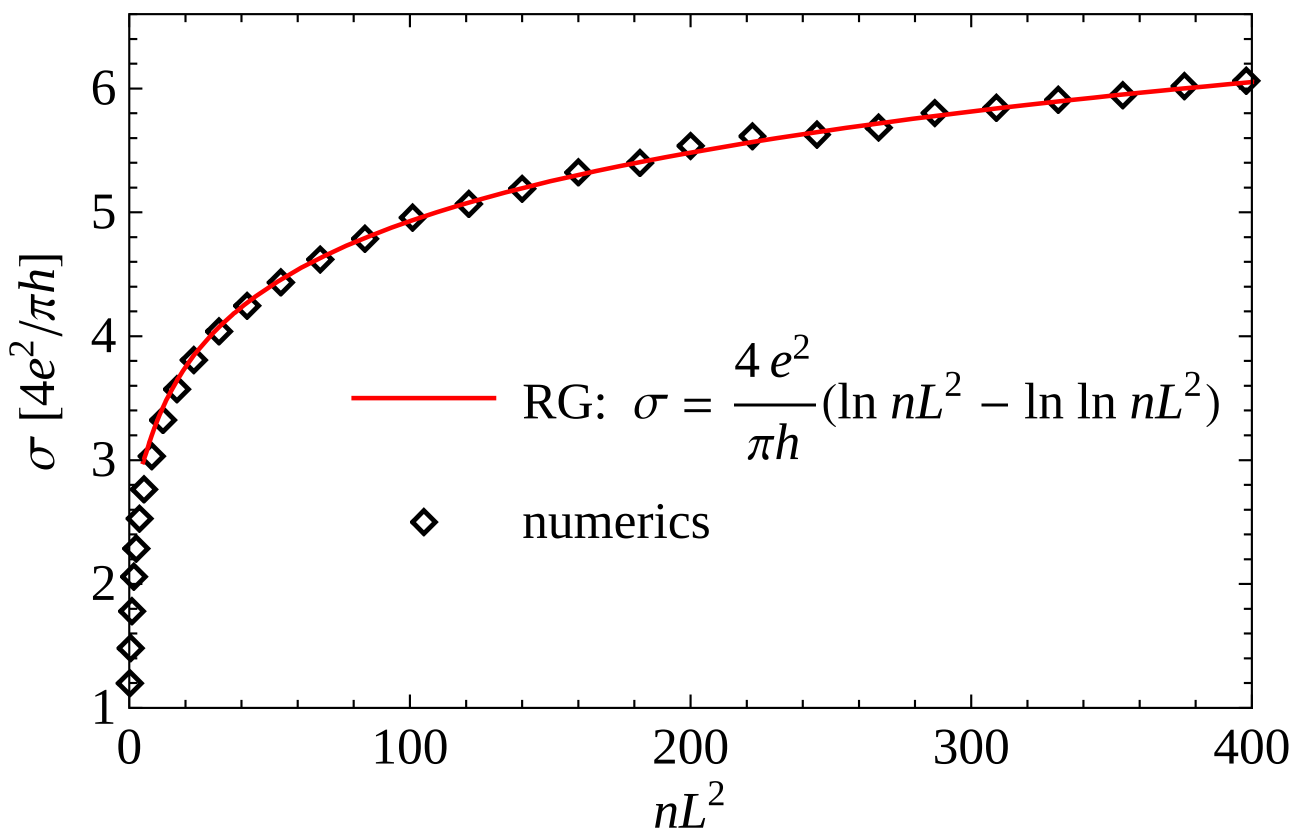


Models of scatterers:

- scalar impurity: smooth on atomic scale (no valley mixing)
- resonant scalar impurity: diverging scattering length, quasi-bound state at the Dirac point
- adatom: on-site potential (valley mixing)
- vacancy: infinitely strong on-site potential

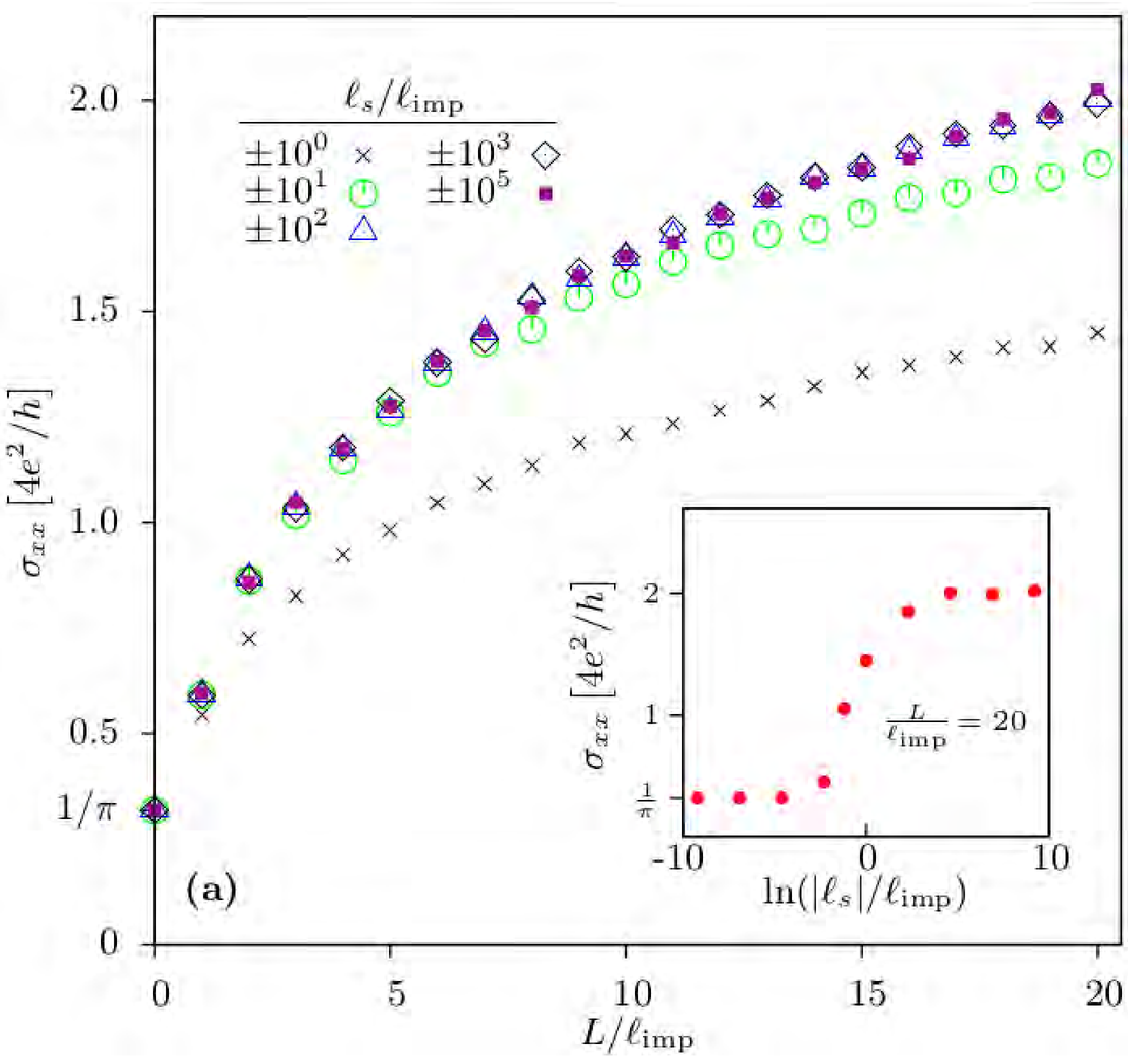
Resonant scalar impurities ($l_s = \infty$)

Ostrovsky, Titov, Bera, Gornyi, ADM, PRL (2010)



- **flow towards supermetal** $\boldsymbol{\sigma \to \infty}$
- **agreement with** $\boldsymbol{\sigma}$ **model RG**

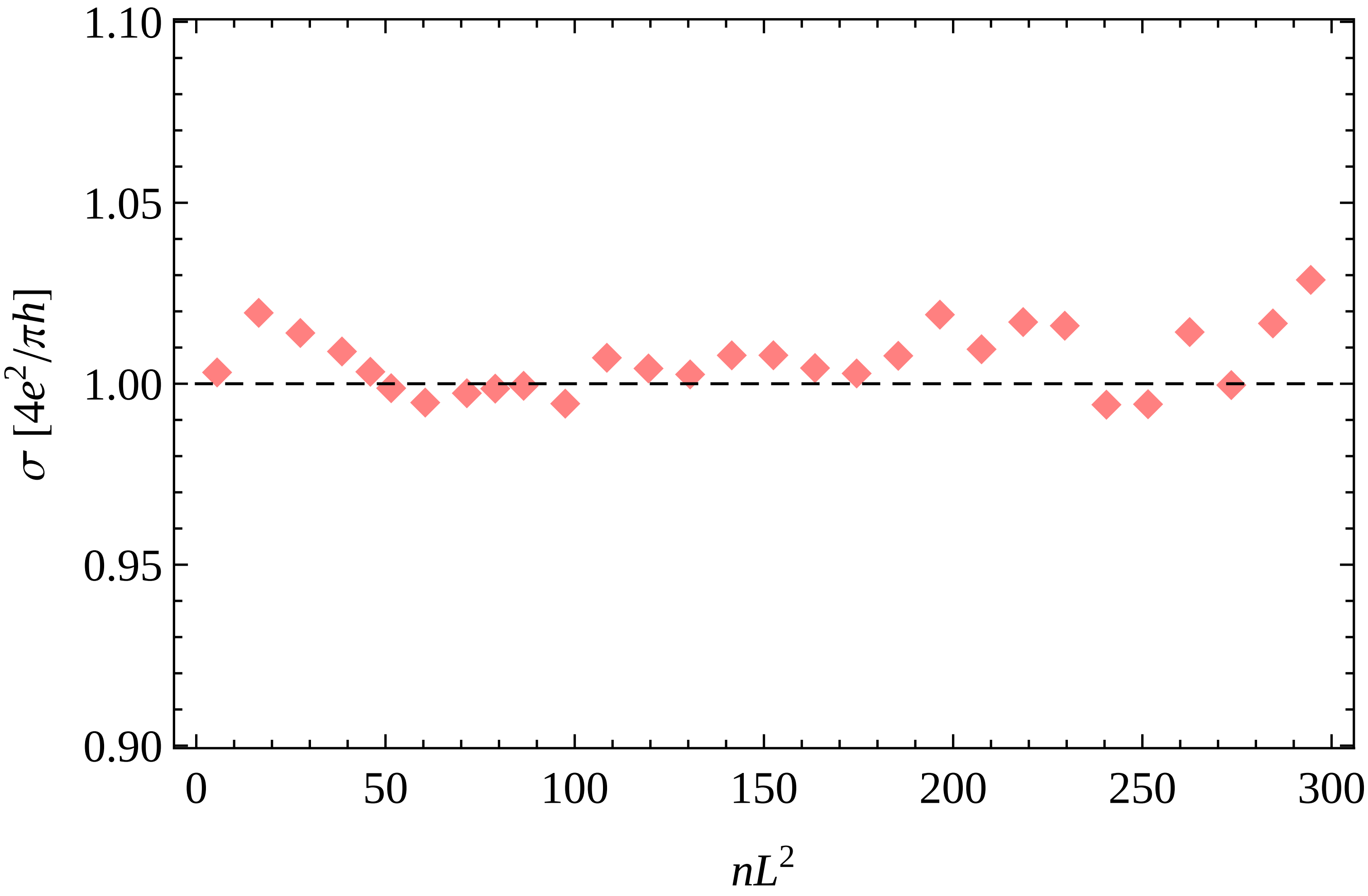
Scalar impurities (finite l_s, random sign)



Large l_s $\longrightarrow$ Symmetry breaking pattern:

DIII (with WZ term) $\longrightarrow$ AII (with $\mathbb{Z}_2$ θ-term)

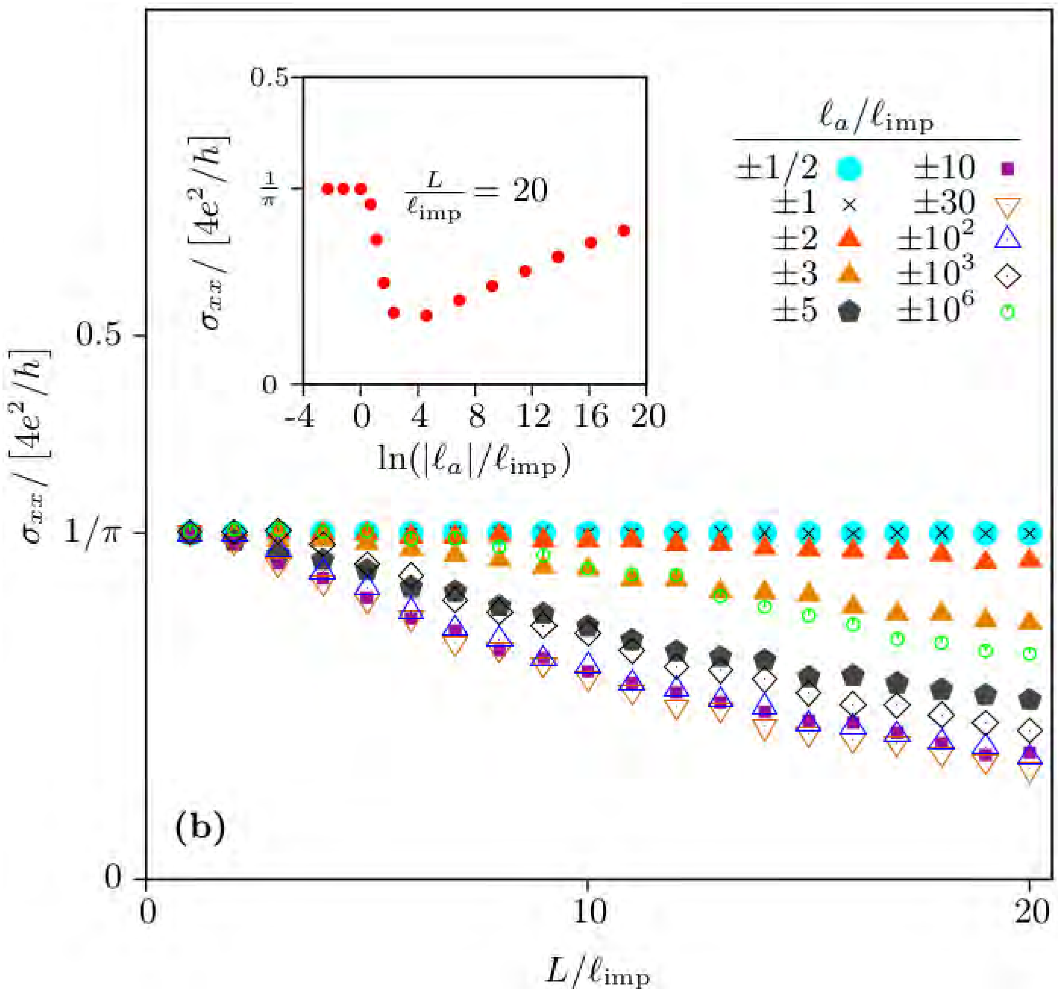
Vacancies



symmetry class BDI (chiral orthogonal)

No localization, $\sigma \to \text{const} \simeq \frac{4}{\pi}\frac{e^2}{h}$

Adatoms (finite l_a, random sign)

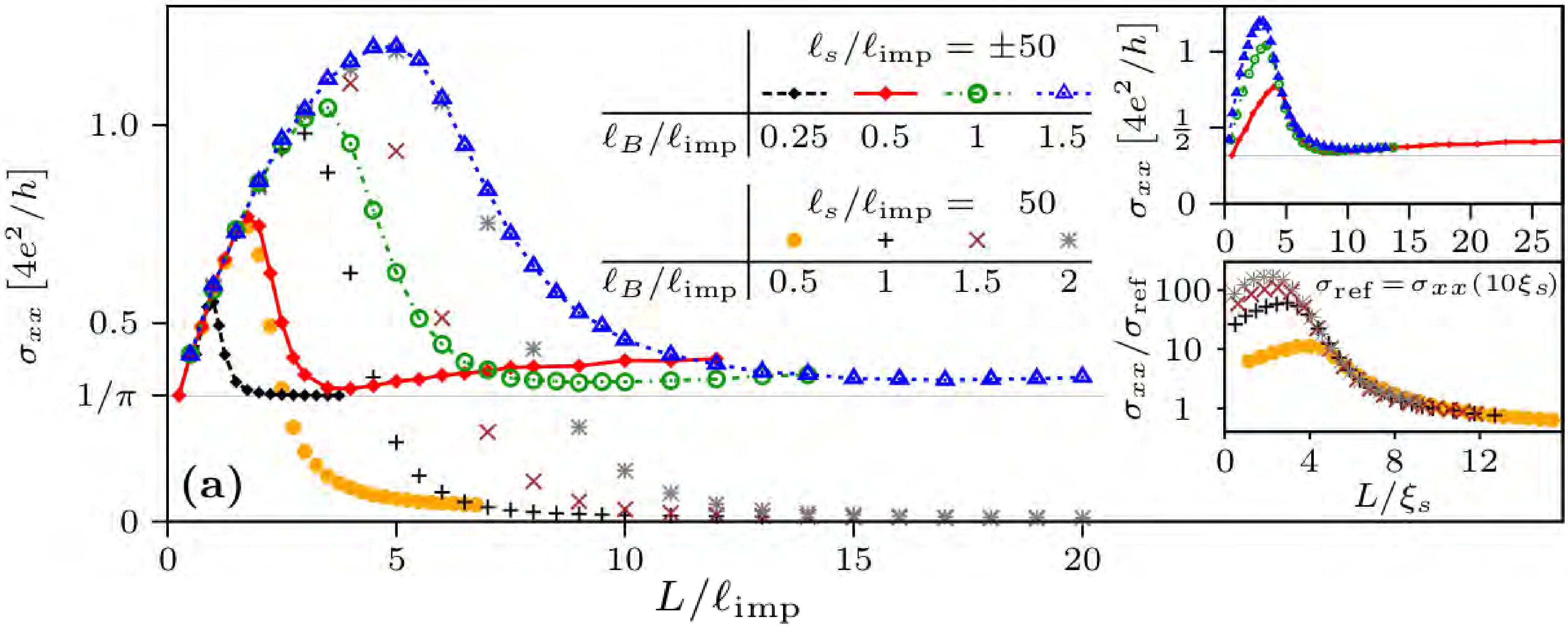


Large $l_a \longrightarrow$ Symmetry breaking pattern: BDI $\longrightarrow$ AI

Vacancies ($l_a \to \infty$): finite conductivity $\sigma \simeq \frac{4}{\pi}\frac{e^2}{h}$ for $L \to \infty$

Localization length ξ – non-monotonous function of l_a

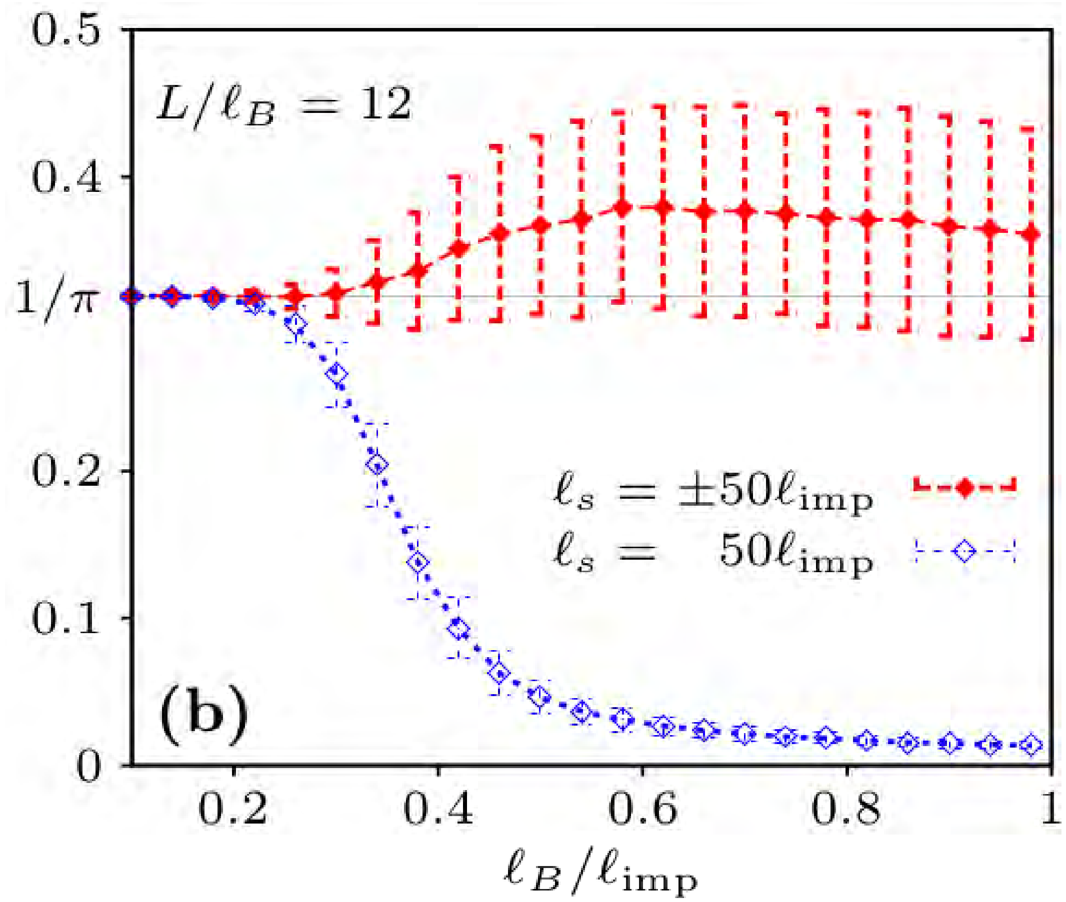
Scalar impurities in magnetic field B

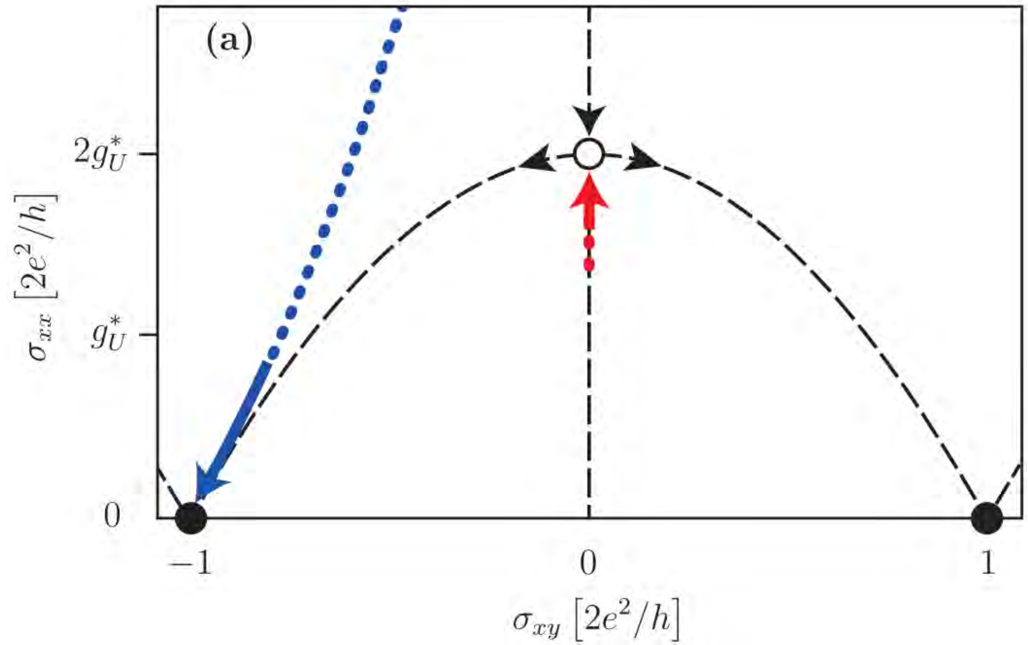


Symmetry breaking pattern: DIII → AII → A for weaker B and DIII → AIII → A for stronger B

Ultimate fixed points: Quantum Hall criticality (random sign of impurity potentials) and localization (fixed sign)

Scalar impurities in magnetic field B

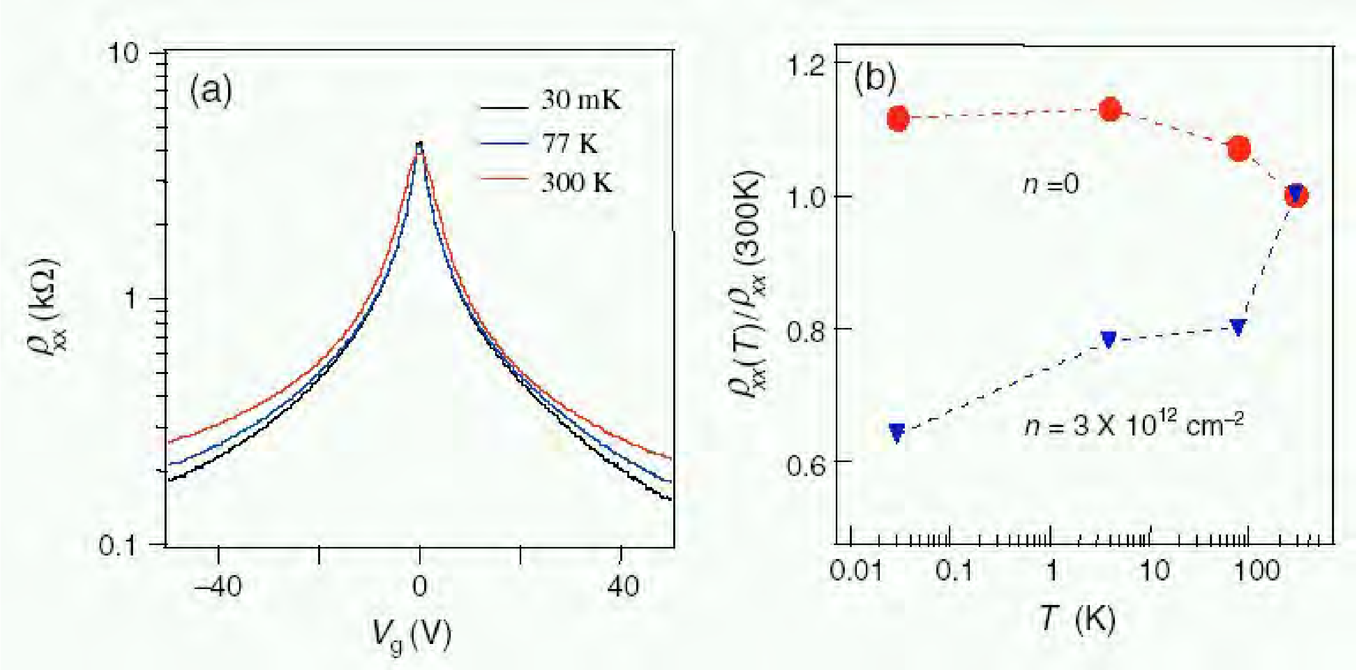


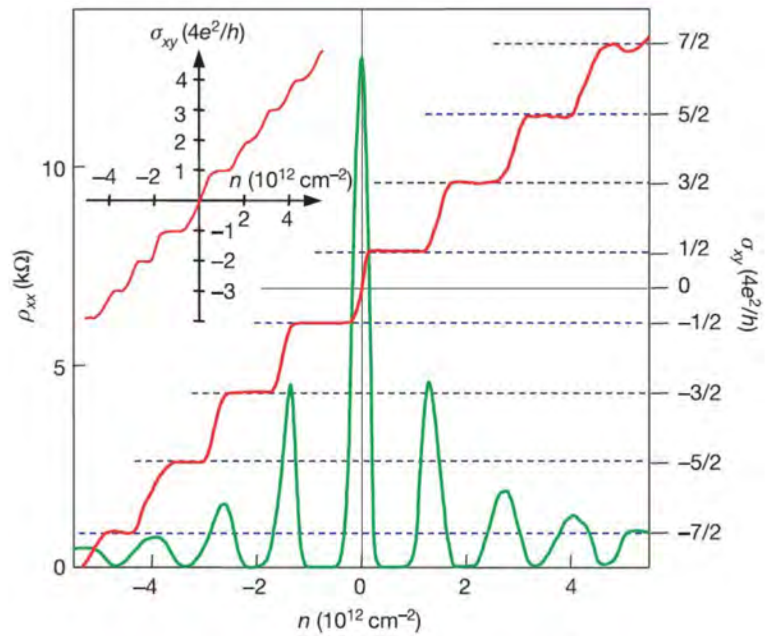


Ultimate fixed points: **Quantum Hall criticality** **(random sign of impurity potentials) and** **localization** **(fixed sign)**

Vertical bars: mesoscopic fluctuations

Graphene: Experiments   Geim-Novoselov and Kim groups





Topological terms explain unconventional properties of high-quality graphene samples:

- absence of localization at Dirac point down to very low temperatures (30 mK), although conductivity $\simeq e^2/h$ per spin per valley

- anomalous QHE: $\sigma_{xy} = (2n+1) \times 2e^2/h$;

QHE transition at $n = 0$ (Dirac point), i.e. at $\sigma_{xy} = 0$

Electron-electron interaction effects

- **Renormalization**

  Virtual processes, energy transfer $\gtrsim T$,
  become stronger when T is lowered

  - **mutual renormalisation of resistivity and interaction,**
  - **zero-T phase diagram and quantum phase transitions**
  - **effect of disorder on superconducting and magnetic instabilities**

- **Dephasing**

  Real inelastic scattering processes, energy transfer $\lesssim T$,
  become weaker when T is lowered

  - **dephasing of quantum interference**
  - **decay of single-particle excitations**
  - **finite-T broadening of localization quantum phase transitions**
  - **finite-T many-body (de-)localization**

Interacting non-linear sigma model (NLσM)

$$S = -\frac{g}{32}\int dr\,\mathrm{Tr}(\nabla Q)^2 + 4\pi T Z_\omega \int dr\,\mathrm{Tr}\,\eta Q$$
$$-\frac{\pi T}{4}\Gamma_s \sum_{\alpha,n}\sum_{r=0,3}\int dr\,\mathrm{Tr}\left[I_n^\alpha t_{r0} Q\right]\mathrm{Tr}\left[I_{-n}^\alpha t_{r0} Q\right]$$
$$-\frac{\pi T}{4}\Gamma_t \sum_{\alpha,n}\sum_{r=0,3}\sum_{j=1}^{3}\int dr\,\mathrm{Tr}\left[I_n^\alpha t_{r} Q\right]\mathrm{Tr}\left[I_{-n}^\alpha t_{r} Q\right]$$
$$-\frac{\pi T}{2}\Gamma_c \sum_{\alpha,n}\sum_{r=0,3}(-1)^r\int dr\,\mathrm{Tr}\left[I_n^\alpha t_{r0} Q I_n^\alpha t_{r0} Q\right]$$

g – conductivity (in e^2/h) $\qquad Z_\omega$ – frequency renormalization

Γ_s , Γ_t, Γ_c – singlet, triplet, and Cooper interaction amplitudes

$Q(\mathrm{r}) = T^{-1}(\mathrm{r})\Lambda T(\mathrm{r})$ – matrix in replica, Matsubara, spin, and p-h

$\Lambda_{nm}^{\alpha\beta} = \mathrm{sgn}\, n\, \delta_{nm}\delta^{\alpha\beta} t_{00} \qquad \eta_{nm}^{\alpha\beta} = n\,\delta_{nm}\delta^{\alpha\beta}t_{00} \qquad (I_k^\gamma)_{nm}^{\alpha\beta} = \delta_{n-m,k}\delta^{\alpha\beta}\delta^{\alpha\gamma}t_{00}$

α, β – replica indices ; $\quad n, m$ – Matsubara indices

$t_{rj} = \tau_r \otimes s_j$ – Pauli matrices in particle-hole and spin spaces

Renormalization group for interacting NLσM in 2D

$$\frac{dt}{dy} = t^2\Big[1 + f(\gamma_s) + 3f(\gamma_t) - \gamma_c\Big]$$

$$\frac{d\gamma_s}{dy} = -\frac{t}{2}(1+\gamma_s)\big(\gamma_s + 3\gamma_t + 2\gamma_c + 4\gamma_c^2\big)$$

$$\frac{d\gamma_t}{dy} = -\frac{t}{2}(1+\gamma_t)\Big[\gamma_s - \gamma_t - 2\gamma_c(1 + 2\gamma_t - 2\gamma_c)\Big]$$

$$\frac{d\gamma_c}{dy} = -2\gamma_c^2 - \frac{t}{2}\Big[(1+\gamma_c)(\gamma_s - 3\gamma_t) - 2\gamma_c^2 + 4\gamma_c^3 + 6\gamma_c\Big(\gamma_t - \ln(1+\gamma_t)\Big)\Big]$$

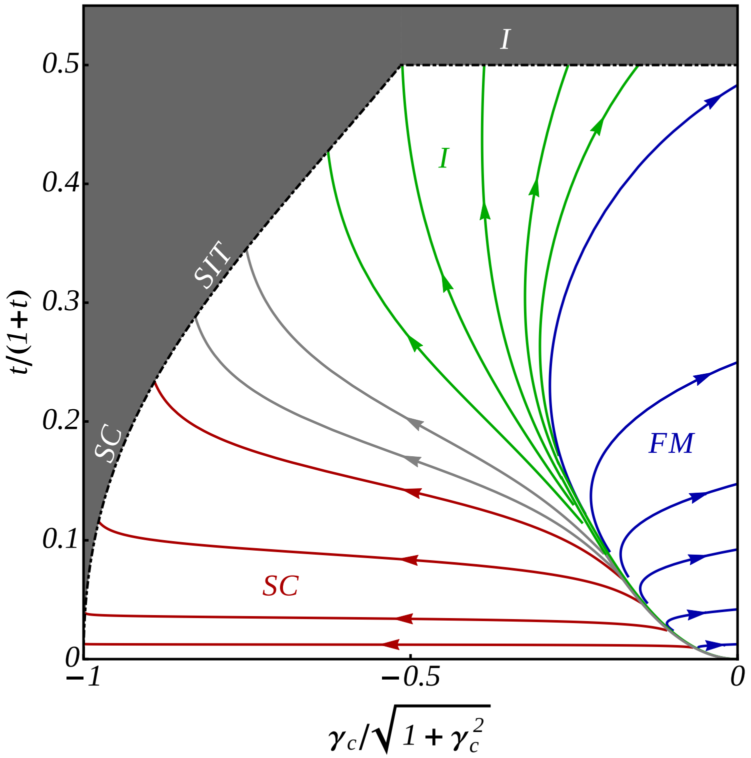
$$\frac{d\ln Z_\omega}{dy} = \frac{t}{2}\Big(\gamma_s + 3\gamma_t + 2\gamma_c + 4\gamma_c^2\Big)$$

$y = \ln(L/l)$ – running RG scale $\qquad f(x) = 1 - (1 + 1/x)\ln(1+x)$

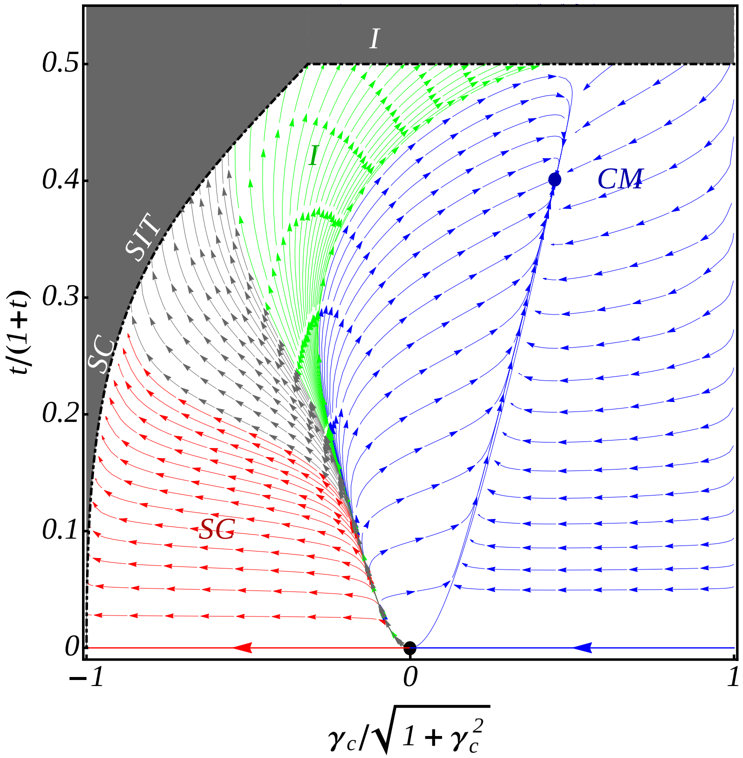
$\gamma_i = \Gamma_i / Z_\omega \qquad t = 2/\pi g$ – dimensionless resistance

Burmistrov, Gornyi, ADM, PRL'12, PRB'15

2D phase diagrams



Coulomb interaction, preserved spin symmetry



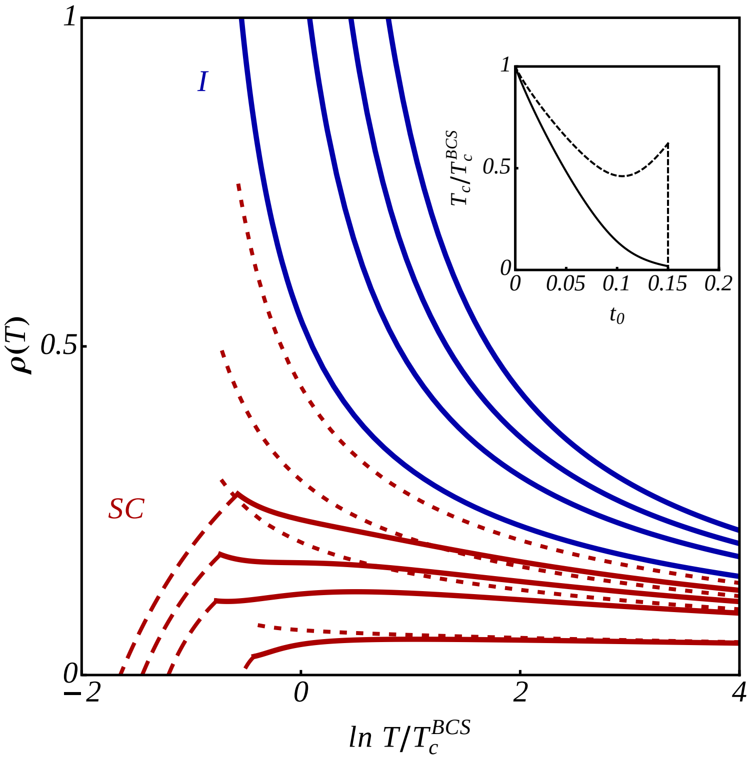
Coulomb interaction, spin-orbit coupling (broken spin symmetry)

I – insulating, SC – superconducting, FM – ferromagnetic, CM – critical metal ($t \sim 1$)

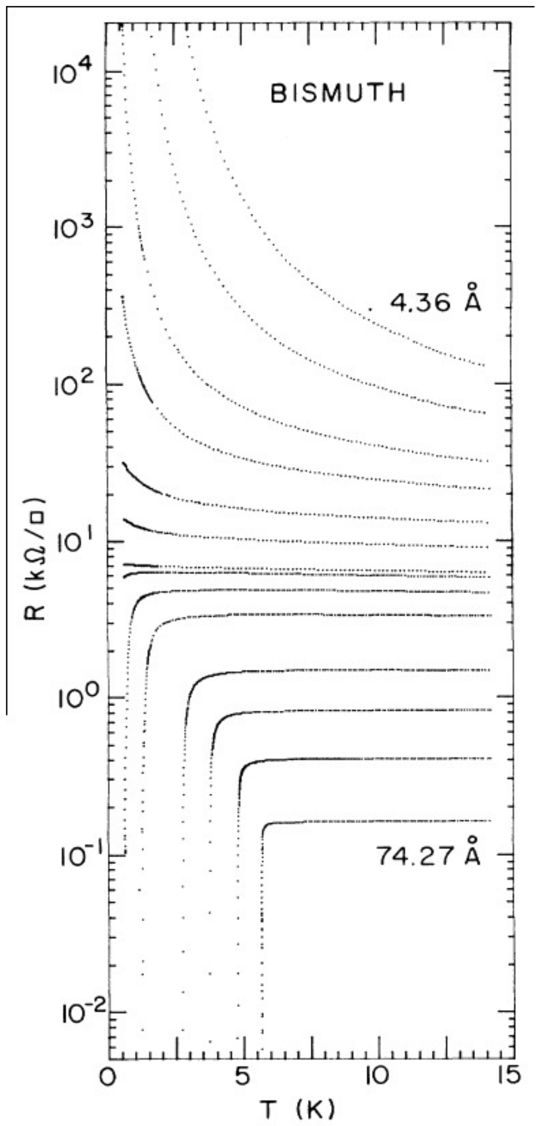
In some cases (SO coupling + short-range interaction or several species) a "supermetal" phase ($t \to 0$) also arises.

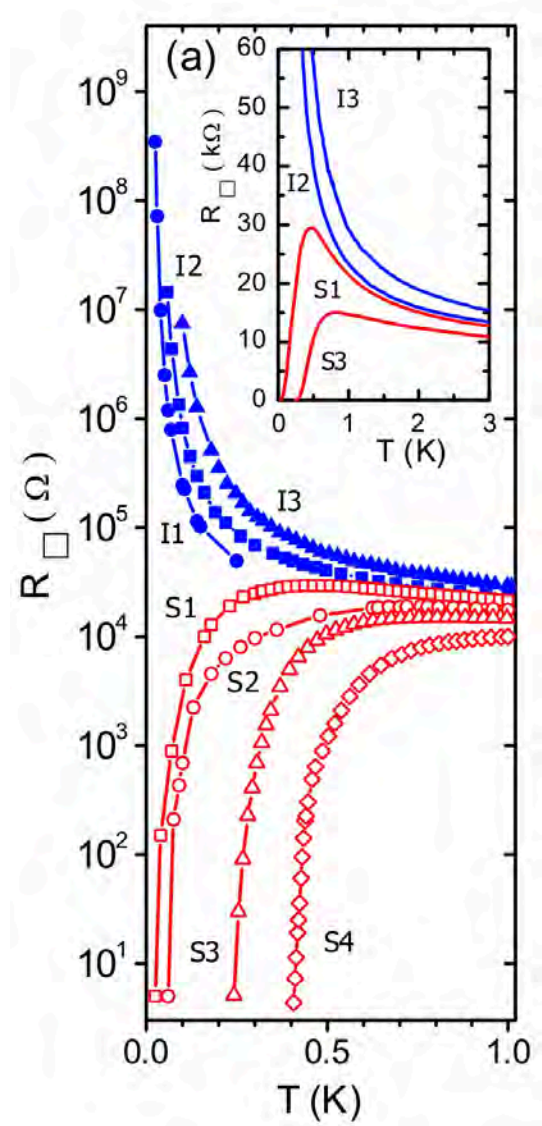
Burmistrov, Gornyi, ADM, PRL'12, PRB'15

T dependence of resistivity across SIT: Coulomb interaction



Burmistrov, Gornyi, ADM, PRL'12, PRB'15





experiment: Bi and Pb films

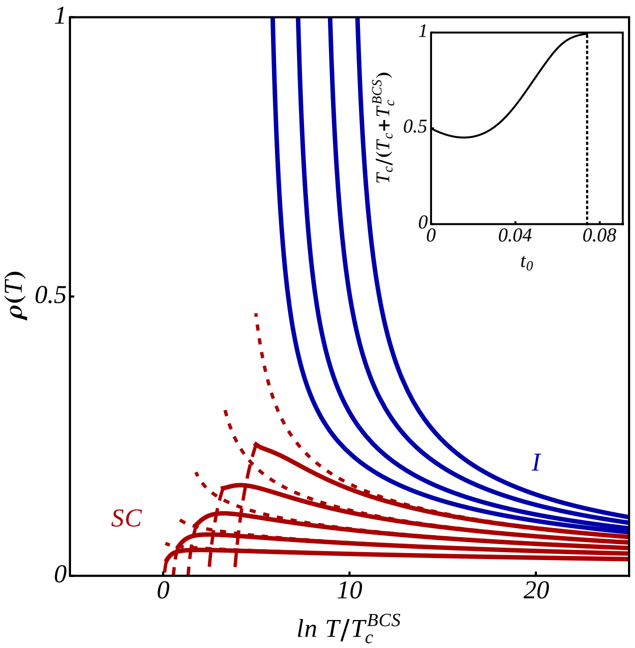
Haviland, Liu, Goldman, PRL'89

exper.: TiN films

Baturina et al, PRL'07

suppression of superconductivity by localization + Coulomb int.

T dependence of resistivity across SIT: Short-range interaction

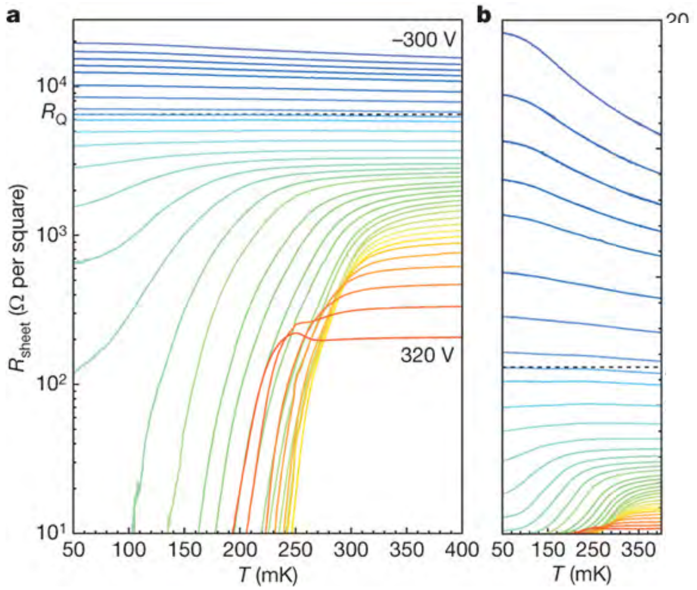


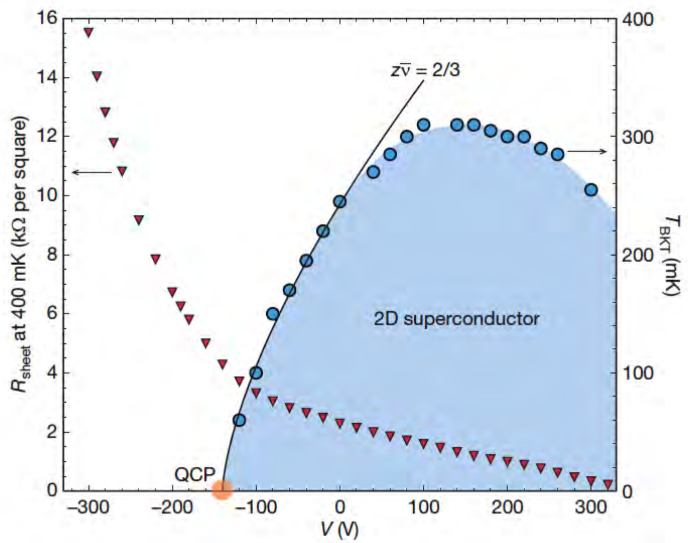
Burmistrov, Gornyi, ADM, PRL'12, PRB'15

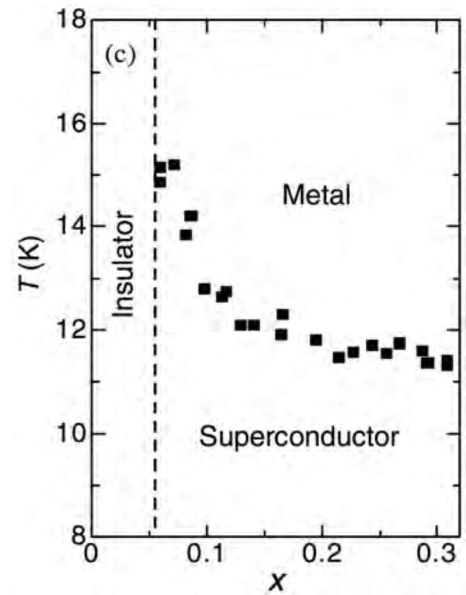
enhancement of superconductivity
by localization !

can be traced back to
multifractality ⟶ renormalisation
towards stronger interaction

Experimental verification ?







Taguchi et al, PRL'06
Li_xZrNCl

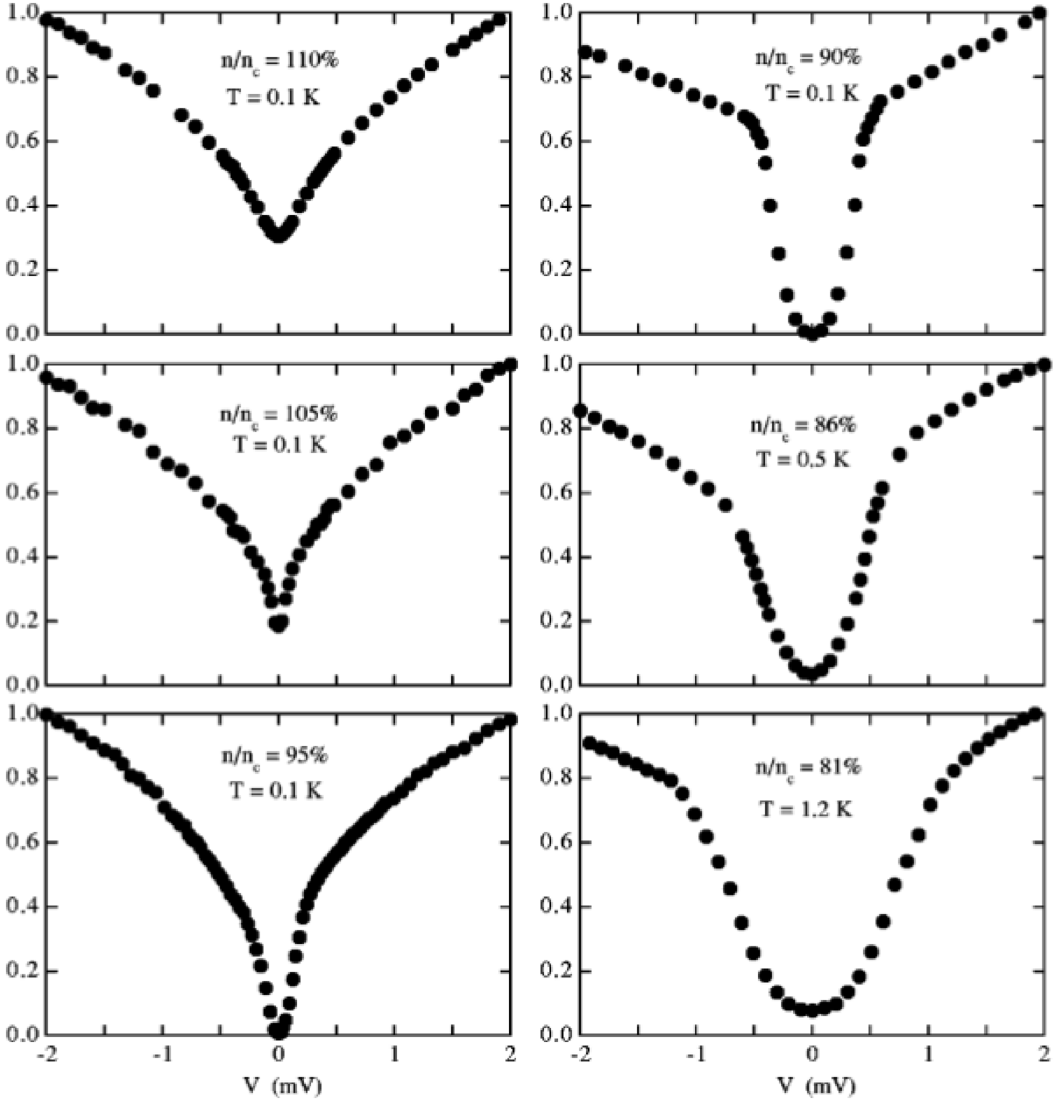
Caviglia et al, Nature'08 $LaAlO_3/SrTiO_3$ interface

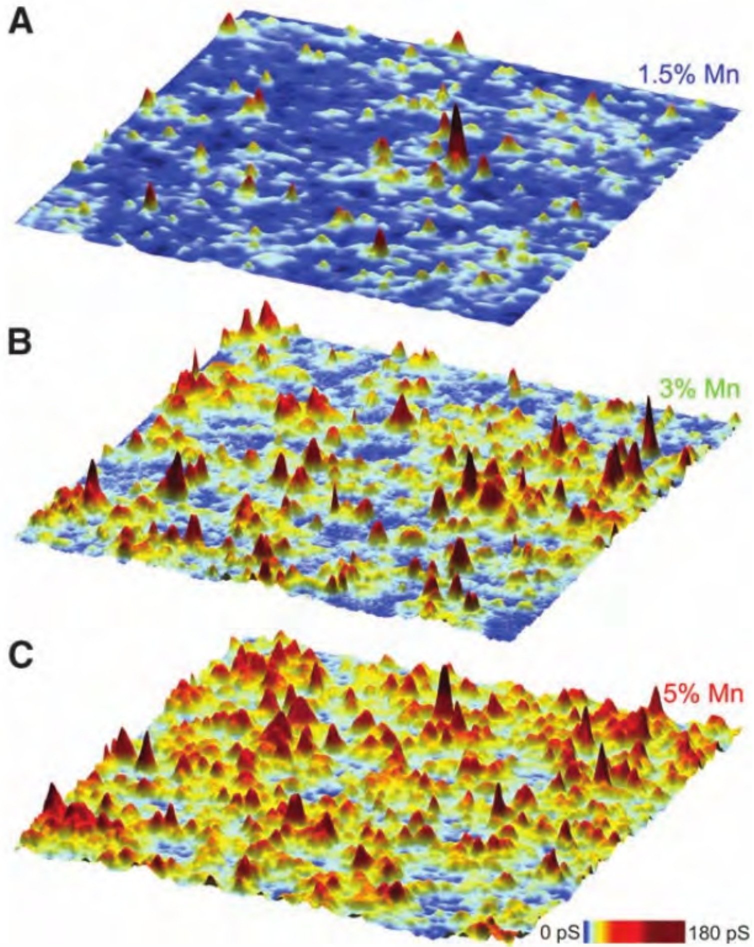
Multifractality and tunneling DOS in systems with Coulomb interaction

Burmistrov, Gornyi, ADM, PRL 111, 066601 (2013);
PRB 89, 035430 (2014)
PRB 91, 085427 (2015)

- Does multifractality survive in the presence of 1/r Coulomb interaction?
- If yes, in what physical observables does it show up?
- What are multifractal exponents in the presence of Coulomb interaction?

TDOS in systems with Coulomb interaction: Experiments





average TDOS, **Si:B**

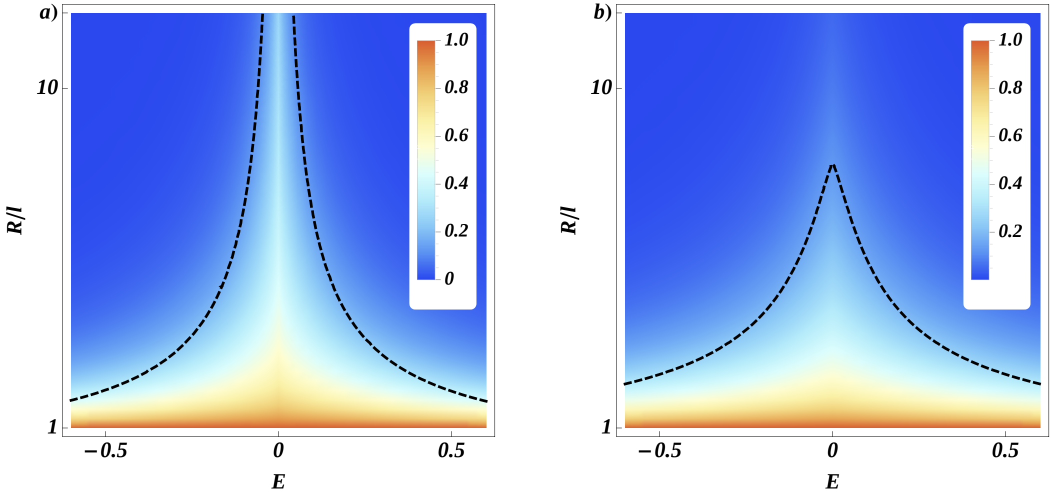
Lee et al, PRB ’99

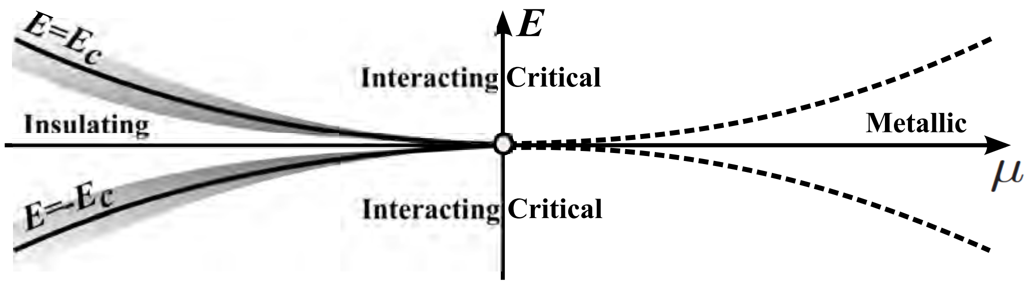
TDOS fluctuations, **$Ga_{1-x}Mn_xAs$**

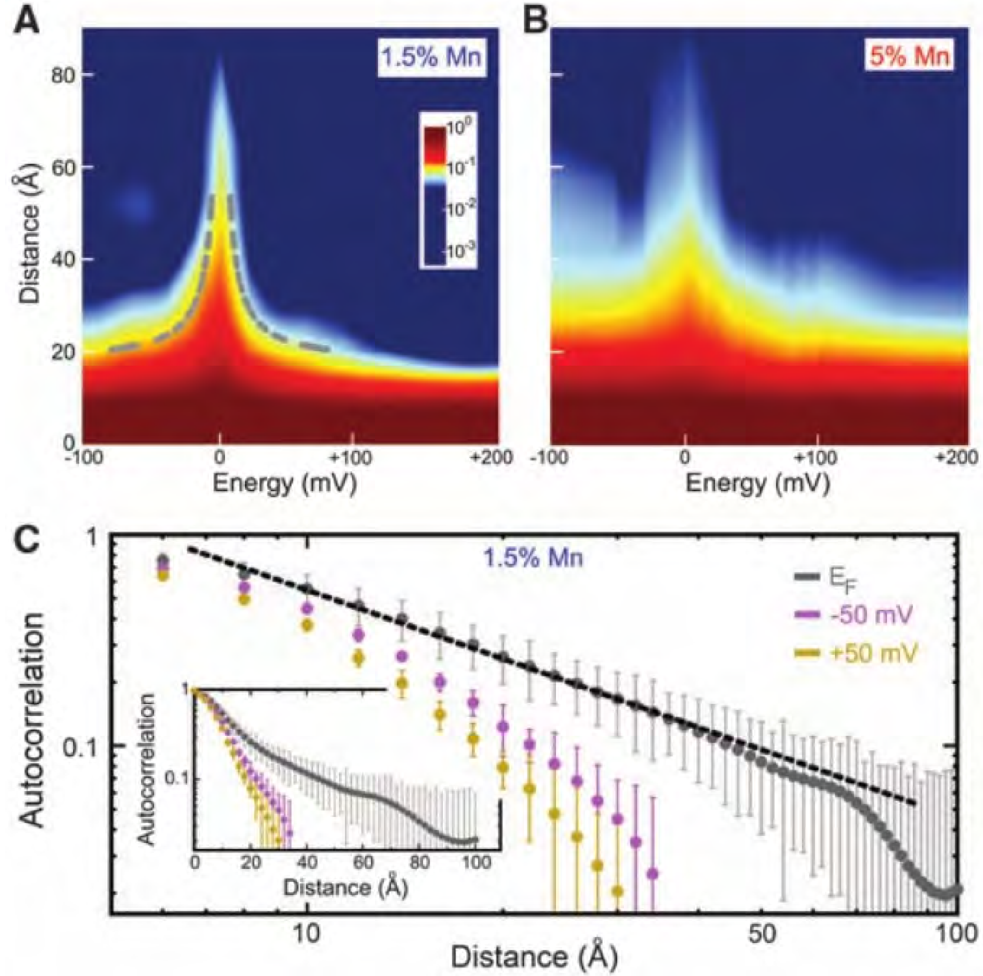
Richardella et al, Science ’10

Multifractal correlations of local TDOS at Anderson transition with Coulomb interaction

$$\langle[\rho(E,r)-\langle\rho(E)\rangle][\rho(E,r+R)-\langle\rho(E)\rangle]\rangle/\langle\langle\rho^2(E)\rangle\rangle$$







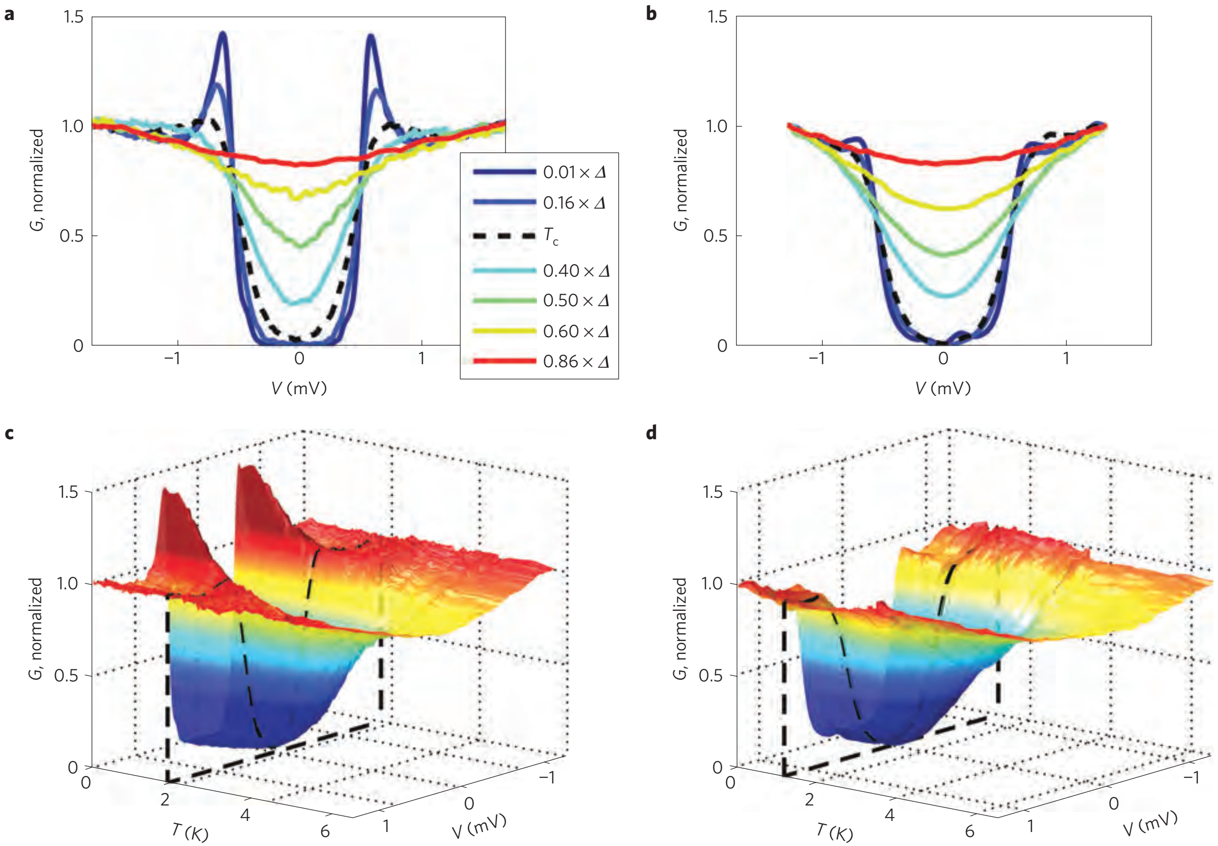
Theory with one-loop exponents

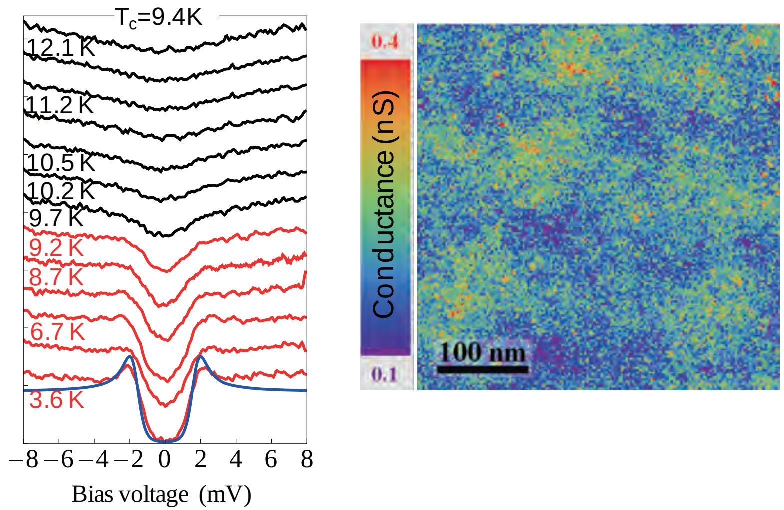
$\nu = 1, \quad z = 1.5, \quad \eta = 0.5$

Experiment

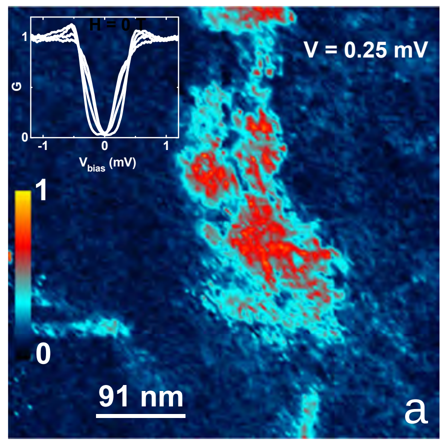
Richardella et al, Science '10

Tunneling spectroscopy near superconductor-insulator transition





Noat et al, PRB 2013, NbN 2-15 nm films



Sacepe et al, Nature Phys. 2011, InO 15-30 nm films

- **Soft gap surviving across SIT and superconductor-metal transition**
- **Strong point-to-point fluctuations of tunneling LDOS**

Kulkarni et al, TiN 5nm films

Sigma-model RG theory: **Burmistrov, Gornyi, ADM, arXiv:1603.03017**

Delocalization by inelastic processes

Problem of "many-body localization":

assume that all single-partcile states are localized
(e.g., 1D or quasi-1D, or a tight-binding model of any d with sufficiently strong disorder)

what happens at finite T (in the absence of external bath)?
Localization, conductivity, other observables – ?

Two opposite limits considered long ago:

Fleishman, Anderson '80: low T (or strong disorder):
Localization in many-body space, zero conductivity

Altshuler, Aronov, Khmelnitskii '82: high T:
dephasing reducing localization to weak-localisation effects,
almost classical conductivity

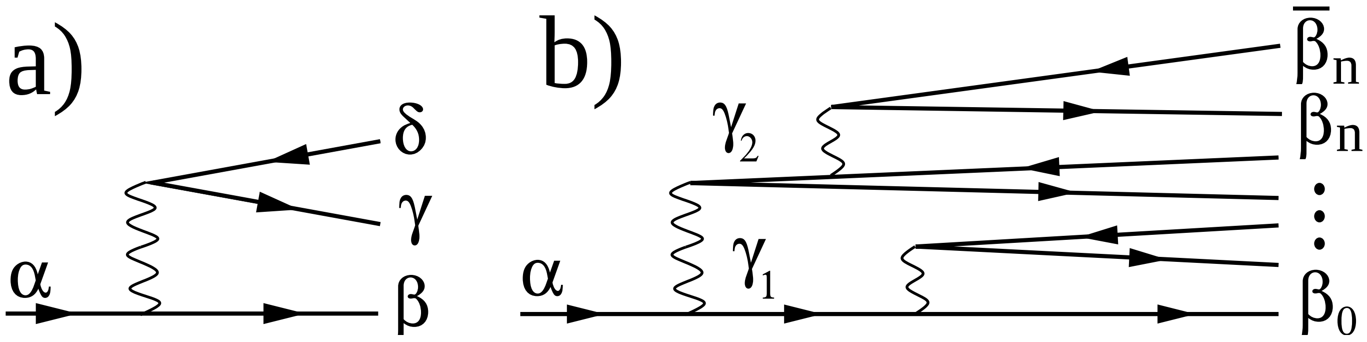
$\longrightarrow$ there should be a transition at an intermediate T
(or intermediate disorder strength at fixed T)

more recently: Gornyi, ADM, Polyakov '05;
Basko, Aleiner, Altshuler '06, ...

Localization in Fock space

Gornyi, ADM, Polyakov '05

Single-particle excitation decay processes:



Lowest order process: $e \to eeh$ $\longrightarrow$ **Golden rule** $\tau_\phi^{-1} \sim V^2/\Delta_\xi^{(3)}$

$V \sim \alpha\Delta_\xi$ – **interaction matrix element**, α - **interaction strength,**

Δ_ξ - **single-particle level spacing in localization volume,**

$\Delta_\xi^{(3)} \sim \Delta_\xi^2/T$ - **three-particle level spacing in localization volume**

Golden rule is justified only if $V > \Delta_\xi^{(3)}$,

which corresponds to $T > T_3$, **where** $T_3 \sim \Delta_\xi/\alpha$

$V < \Delta_\xi^{(3)}$ $\longrightarrow$ **no transition on the Golden Rule level**

Localization in Fock space and MIT

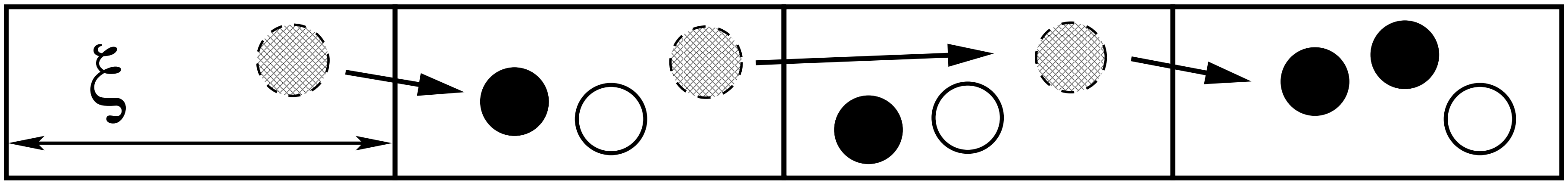
Gornyi, ADM, Polyakov '05

Higher orders? $\longrightarrow$ **have to analyze** $V^{(n)}/\Delta^{(2n+1)}$

$$V^{(n)} = \sum_{\text{diagrams}} \sum_{\gamma_1,\cdots\gamma_{n-1}} V_1 \prod_{i=1}^{n-1} \frac{V_{i+1}}{E_i - \epsilon_{\gamma_i}}$$

$\longrightarrow$ **optimal paths: "ballistic":**

a "string" with a few excitations per localization volume



$$\frac{V^{(n)}}{\Delta^{(2n+1)}} \sim \left(\frac{T}{T_3}\right)^n$$

$\longrightarrow$ **Localization transition at** $T = T_c \sim T_3$

Mapping onto Bethe lattice

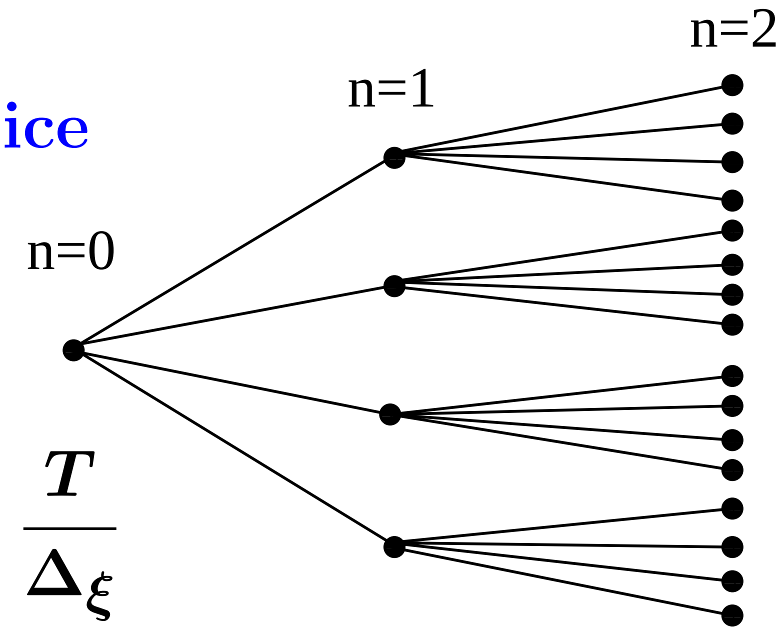
Gornyi, ADM, Polyakov '05

Interacting problem in Fock space

$\longrightarrow$ Anderson model on the Bethe lattice

$\longrightarrow$ Metal-Insulator Transition at

$$\Delta/V = 4 \ln K$$



K: Coordination number $K \sim \frac{\Delta_\xi}{\Delta_\xi^{(3)}} \sim \frac{T}{\Delta_\xi}$

Δ: Level spacing of $n = 1$ states: $\Delta = \Delta_\xi^{(3)}$

V: hopping matrix element:
interaction matrix element $V \sim \alpha \Delta_\xi$

$\longrightarrow$ transition temperature $$T_c = \frac{\Delta_\xi}{\alpha \ln \alpha^{-1}}$$

Basko, Aleiner, Altshuler '06: same result from self-consistent Born approx.

Summary

- Anderson localization: basic properties, field theory
- Wave function multifractality
- Symmetries of disordered systems
- Manifestations of topology in localization theory
- Influence of electron-electron interaction

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